Blowing Up and Down in 4-Manifolds

By

Paul Michael Melvin

A.B. (Haverford College) 1971

DISSERTATION

Submitted in partial satisfaction of the requirements for the degree of

DOCTOR OF PHILOSOPHY

in

Mathematics

in the

GRADUATE DIVISION

of the

UNIVERSITY OF CALIFORNIA, BERKELEY

Approved: John R. Stallings

[illegible]

Robion C. Kirby

Committee in Charge

Blowing Up and Down in 4-Manifolds

By

Paul Michael Melvin

Abstract

This paper is concerned with the topology of 4-dimensional manifolds. In particular we are interested in links of 2-spheres in simply connected 4-manifolds.

Our motivating question is whether or not there exist inequivalent unit links (all components with self intersection ± 1) which are concordant. A positive answer would provide a counterexample to the 5-dimensional h-cobordism conjecture.

Our principal result states that any two unit links which can be joined by a concordance with simply connected levels must be equivalent.

The proof uses the technique of blowing up points and blowing down spheres in 4-manifolds, together with some results on extending diffeomorphisms of 3-manifolds. The underlying approach is "constructive" handlebody theory.

As an additional result, unrelated to our main line of thought, we show that the 3-dimensional oriented bordism group of diffeomorphisms vanishes.

Finally, we apply our results to show that certain homotopy 4-spheres arising naturally from knots of 2-spheres in S^4 are standard.

Robion C Kirby

Table of Contents

	Acknowledgements	ii
1.	Definitions and Outline of Results	1
2.	Concordance and h-Cobordism	10
3.	Extending Diffeomorphisms of 3-Manifolds	14
4.	A Digression	37
5.	0-Concordance	42
6.	Embedding CP^1 in CP^2 and Gluck's Construction	54
7.	Handlebody Structure for τS	60
	References	68

Acknowledgements

I would like to express my gratitude to my advisor Rob Kirby for showing me his refreshing outlook on mathematics, for sharing so many ideas, and for giving endless support and encouragement during my work on this thesis.

Thanks also go to John Harer and Steve Kaplan for many enlightening discussions, and to Margaret Rowell for wonderful voyages to other worlds.

Finally, my warmest thanks go to Barbara Mindell for suffering and rejoicing with me.

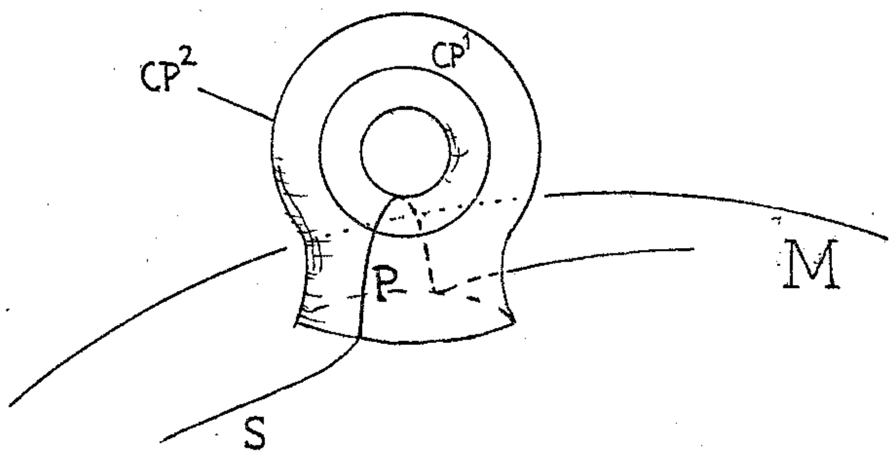
§1. Definitions and Outline of Results

The general purpose of this paper is to study simply connected 4-manifolds from the point of view of links of 2-spheres in 4-manifolds. Our principal tool will be the technique of blowing up. This we define below in a purely topological way, ignoring the complex structure that comes into play in the usual definitions from algebraic geometry.

Our setting will be the smooth category, unless specified otherwise. We write ∂M for the boundary of a (smooth) manifold M. If M is oriented, then $-M$ will denote the same manifold with the opposite orientation. For any pair of oriented manifolds M and N, and any orientation reversing diffeomorphism $h:\partial N \longrightarrow \partial M$, we may form a closed manifold $M \underset{h}{\smile} N$ from the disjoint union of M and N by identifying points in ∂N with their images under h in ∂M. This has a unique smooth structure compatible with the inclusions $M \rightarrow M \underset{h}{\smile} N \leftarrow N$.

There are two ways of blowing up a point x in a 4-manifold M (replacing a 4-ball about x by a Hopf disc bundle with one of the two possible orientations). These yield the connected sum of M near x with either CP^2 or $-CP^2$. We assume that the connected sum is taken away from the standard $\pm CP^1$ ($z_2 = 0$ in homogeneous coordinates $[z_0, z_1, z_2]$ in $\pm CP^2$), which will be called the <u>image</u> of x.

If S is a 2-manifold in M, then we may blow up a point <u>on S</u> by taking the connected sum pairwise with $(\pm CP^2, P)$, where P is a projective line cutting $\pm CP^1$ in one point (e.g. $z_1 = 0$). $S \# P$ is then called the <u>image</u> of S.



We may invert this construction. Let T be a link of 2-spheres in the interior of M all of whose components have self intersection ± 1. Such a link will be called a <u>unit link</u>. By M/T we denote the 4-manifold gotten from M by <u>blowing down</u> L (replacing a tubular neighborhood of each 2-sphere S in T by a 4-ball — the center of this ball is called the <u>image</u> of S). M/T is well defined since $\Gamma_4 = 0$ [Cerf]. The reader is cautioned not to confuse M/T with the quotient space of M obtained by identifying T to a point. Here we identify each component of T to a different point.

Note that $M - T$ sits naturally in M/T as the complement of a collection of points, the images of the components of T. Blowing up these points in M/T appropriately yields M. In particular

$$M = M/T \# p(CP^2) \# q(-CP^2)$$

where p is the number of components of T with self intersection $+1$ and q is the number with self intersection -1.

Similarly if we first blow up a collection of points in M and then blow down their images, then we get M back again.

We call two links T_0 and T_1 in M <u>equivalent</u> if there is a diffeomorphism of M carrying one onto the other. It is clear that M/T_0 and M/T_1 are diffeomorphic if T_0 and T_1 are equivalent unit links. Conversely, if M/T_0 and M/T_1 are diffeomorphic, then by homogeneity there is a diffeomorphism carrying $M - N_0$ to $M - N_1$, where N_i is an open tubular neighborhood of T_i, $i = 0, 1$. Since $\Gamma_4 = 0$, this extends to a diffeomorphism of M carrying T_0 to T_1. Thus for unit links, T_0 and T_1 are equivalent if and only if $M/T_0 = M/T_1$.

Two links T_0 and T_1 in M are <u>concordant</u> if there is a proper embedding $f: T\times I \longrightarrow M\times I$ with $f(T\times i)$ and $T_i \times i$ equivalent in $M\times i$, $i = 0, 1$. Here I denotes the unit interval $[0, 1]$ and proper means that $f^{-1}(\partial(M\times I)) = \partial(T\times I)$. Note that we are using concordance in a weaker sense than usual as M may have diffeomorphisms which are not isotopic to the identity.

For unit links, concordance has the following interpretation

Theorem 2.1 If T_0 and T_1 are unit links in a closed, simply connected 4-manifold M, then T_0 and T_1 are concordant if and only if M/T_0 and M/T_1 are h-cobordant.

The proof will be given in §2.

This link setting applies to any h-cobordant, closed, simply connected 4-manifolds M_0 and M_1. For it follows from [Wall] that M_0 and M_1 become diffeomorphic after blowing up sufficiently many points in each. Thus there are unit links T_0 and T_1 in some closed, simply connected 4-manifold M such that $M_i = M/T_i$, $i = 0, 1$. Hence Theorem 2.1 shows that the 5-dimensional h-cobordism conjecture is equivalent to

Conjecture 1 Concordant unit links in a closed, simply connected 4-manifold are equivalent.

In searching for counterexamples we need only consider unit links in connected sums of CP^2 and $-CP^2$, for we may insure that the common manifold M, obtained as above by blowing up points in two h-cobordant manifolds M_0 and M_1, is such a manifold. In effect we first blow up two points with opposite orientations, obtaining h-cobordant manifolds $M_i \# CP^2 \# -CP^2$, $i = 0, 1$, with isomorphic odd, indefinite intersection forms. By the classification of integral, unimodular, symmetric bilinear forms, these forms are isomorphic to some

$$\langle 1\rangle \oplus \cdots \oplus \langle 1\rangle \oplus \langle -1\rangle \oplus \cdots \oplus \langle -1\rangle$$

[Husemöller - Milnor] and so the $M_i \# CP^2 \# -CP^2$ are also h-cobordant to a connected sum of copies of CP^2 and $-CP^2$ [Wall]. As above, blowing up extra points if necessary, we obtain M and unit links T_i with $M/T_i = M_i$, $i = 0, 1$, where M is a connected sum of copies of CP^2 and $-CP^2$.

At the end of §2, we show that the notions of concordance, homotopy, and homology are all equivalent for unit links in a closed, simply connected 4-manifold M. In particular, if the homology classes represented by the components of two unit links T_0 and T_1 are the same, then M/T_0 and M/T_1 are h-cobordant. Thus, for example, the existence of two inequivalent but homologous unit knots in a closed, simply connected 4-manifold would yield a counterexample to the 5-dimensional h-cobordism conjecture. Conversely, any counterexample which stabilizes after forming the connected sum with one $\pm CP^2$ must arise in this way.

On the positive side we have Theorem 5.2 below. First we need a definition. Note that any concordance $f: T\times I \to M\times I$ may be adjusted by a small isotopy (relative to the boundary) so that qf is a Morse function, where $q: M \times I \longrightarrow I$ is projection. Such a concordance will be called generic. Let the genus of f be the maximal genus of the components of the 2-manifolds $(qf)^{-1}(t)$, where t ranges over all regular values of qf. Let n be some non-negative integer.

Definition Two links in a 4-manifold are n-concordant if there is a generic concordance of genus n between them.

Note that concordant links in a 4-manifold are n-concordant for some n. Thus the following result may be viewed as the first step in a proof of the h-cobordism conjecture.

Theorem 5.2 0-concordant unit links in a 4-manifold are equivalent.

In §3 we give some preliminary results on extending diffeomorphisms of closed 3-manifolds to 4-manifolds which they bound.

We digress in §4 (using the point of view of §3) to give a proof of the vanishing of the 3-dimensional oriented bordism group of diffeomorphisms.

In §5 we give the proof of Theorem 5.2 using the tools developed in §3.

We now specialize to the case of unit links in CP^2

(or equivalently in $-CP^2$, as orientation makes no difference in our considerations) where any unit link has exactly one component S (homologous to the standardly embedded CP^1). It is easy to see that blowing down S in CP^2 yields a homotopy 4-sphere Σ. Σ is h-cobordant to S^4 [Wall] so Theorem 2.1 shows that S is concordant to CP^1. Thus for CP^2 Conjecture 1 reads

Conjecture 2 Any unit knot in CP^2 is equivalent to CP^1 (cf. Problem 4.23 in [Kirby$_2$]).

A discussion of some conjectures related to Conjecture 2 will be given in §6. In particular we will consider the special case of knots in CP^2 which intersect CP^1 in exactly one point. We show that Conjecture 2 in this case is equivalent to a problem of Herman Gluck about knotted 2-spheres in S^4.

Precisely, let $\rho(\theta)$ denote the diffeomorphism of S^2 which rotates S^2 about its polar axis through an angle of $\theta \in S^1$. Thus $\rho: S^1 \to SO(3)$ represents the nontrivial element of $\pi_1 SO(3)$. Given a 2-sphere S in S^4, choose an embedded $S^2 x B^2$ with $S^2 x 0 = S$ and define

$$\tau S = (S^4 - S^2 x \mathrm{int}(B^2)) \underset{\tau}{\cup} S^2 x B^2$$

where the map $\tau: S^2 x S^1 \to S^2 x S^1$ is given by $\tau(s, \theta) = (\rho(\theta)(s), \theta)$. Gluck shows that τS is a well defined homotopy 4-sphere and asks

Question Is $\tau S = S^4$? [Gluck]

An affirmative answer is known for spun knots [Gluck] and more generally twist spun knots [Gordon].

We shall prove

Proposition 6.2 If S is a knot in S^4 and S' is the unit knot in CP^2 obtained by blowing up a point on S, then $CP^2/S' = \tau S$.

Note that any knot in CP^2 which intersects CP^1 in one point arises in this way.

As a corollary to the Proposition and to Theorem 5.2 we have $\tau S = S^4$ for knots S in S^4 which are 0-concordant to the unknot.*

In fact the proof of Theorem 5.2 will show more. We define two links T_0 and T_1 to be cobordant if there is a 3-manifold N properly embedded in $M \times I$ with $N \cap M \times i = T_i$. There is as above the notion of n-cobordant links. The 0-cobordism classes of 2-spheres in S^4 form a semigroup C under connected sum. Let H denote the semigroup of homotopy 4-spheres Σ, under connected sum,

---

* A subclass of these knots called ribbon knots has been in [Yajima] and [Yanagawa]. These knots can in fact be spun (in the sense of [Gluck]) and are thus determined by their complements.

for which $\Sigma \# CP^2 = CP^2$. Then we have

Theorem 6.3 τ defines a semigroup homomorphism $\tau : C \to H$.

Corollary 6.4 If S is a knot in S^4 which is invertible in C, then τS is homeomorphic to S^4.

In §7 we will give an alternate approach to Gluck's question in terms of handlebody theory. We show, under certain severe restrictions on the critical points of an embedded 2-sphere S in S^4, that τS can be built without 3-handles. This reduces the question of the homeomorphism type of τS to an algebraic problem.

§2. Concordance and h-Cobordism

First we generalize the notions of blowing up and down to concordances of points and links in 5-manifolds.

Let W be a 5-manifold with non-empty boundary. A concordance of a closed k-manifold C_0 in W is an embedding $f: C_0 \times I \longrightarrow W$ which is proper $(f^{-1}(\partial W) = C_0 \times \partial I)$. The image of f is a proper $(k+1)$-submanifold C of W, also sometimes called the concordance.

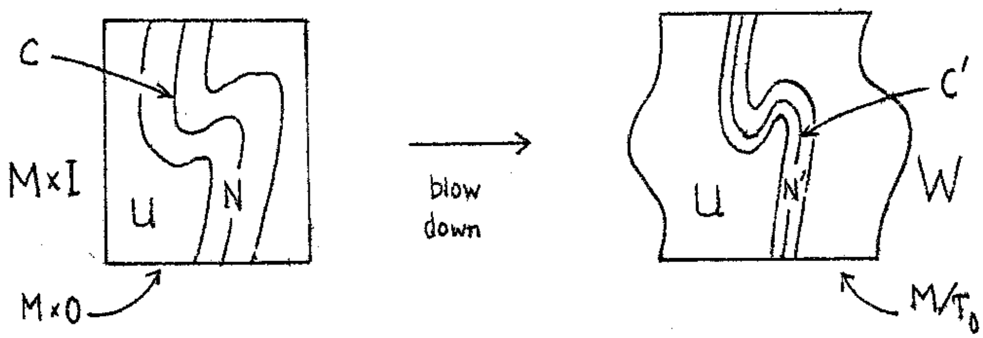
We proceed as in §1, crossing our constructions with the unit interval.

If ∂C is a finite collection of points in ∂W $(k = 0)$ we may blow up C by replacing a tubular neighborhood of each arc in C with $H \times I$ attached along $\partial H \times I$, where H is the Hopf disc bundle with either orientation. Once we choose orientations, this is well defined since $\Gamma_4 = 0$ [Cerf] and $\Gamma_5 = 0$ [Smale].

If ∂C is a unit link in ∂W $(k = 2)$ then we may form a 5-manifold W/C by blowing down C (replacing a tubular neighborhood of each component of C by a 5-ball $B^4 \times I$ attached along $S^3 \times I$). W/C is well defined, as above, and $\partial(W/C) = \partial W/\partial C$. If we view $W - C$ as the complement in W/C of a concordance of points, C', then W is obtained from W/C by blowing up C' appropriately.

<u>Theorem 2.1</u> If T_0 and T_1 are unit links in a closed, simply connected 4-manifold M, then T_0 and T_1 are concordant if and only if M/T_0 and M/T_1 are h-cobordant.

<u>Proof</u> First suppose that T_0 and T_1 are concordant. Let U denote the complement in $M\times I$ of an open tubular neighborhood N of this concordance C. Blowing down C we obtain a simply connected 5-manifold W with boundary $M/T_0 \cup -M/T_1$. Let C' denote the resulting concordance of points. Then $N' = W - U$ is an open tubular neighborhood for C' in W.



The relative Mayer Vietoris sequence for $(N, N \cap M\times 0)$ and $(U, U \cap M\times 0)$ in $M\times I$ (cf. p.187 in [Spanier]) shows that $H_*(U, U \cap M\times 0) = 0$. The corresponding sequence in W for $(N', N' \cap M/T_0)$ and $(U, U \cap M/T_0)$ now gives $H_*(W, M/T_0) = 0$. It follows from theorems of Hurewicz and Whitehead that W is an h-cobordism.

Conversely, suppose W is an h-cobordism between M/T_0 and M/T_1. For $i = 0, 1$, let p_i and q_i denote the number of components of T_i with self intersection $+1$ and -1, respectively. It is clear from rank and signature considerations that $p_0 = p_1$ and $q_0 = q_1$. Thus there is a concordance in W between the points in the image of T_0 and those in the image of T_1, which when blown up appropriately yields a 5-manifold V with boundary $M \cup -M$ and a concordance in V between T_0 and T_1. As above we see that V is an h-cobordism. It follows from [Barden] that V must be diffeomorphic to $M\times I$, and so T_0 and T_1 are concordant.

Note (June '77) does not apply unless every homotopy equivalence $M \to M$ is homotopic to a diffeomorp (true for $M = \mathbb{C}P^2$, false for $M = S^2 \times S^2$! can get down using stab Norms in the [illegible])

Remark 2.2 If T_0 and T_1 are concordant links, then they are homotopic, up to equivalence. In effect, if $f: T\times I \longrightarrow M\times I$ is the concordance and $p: M\times I \longrightarrow M$ is projection, then pf is a homotopy between T_0' and T_1' (where T_i' is equivalent to T_i).

Furthermore it is clear that homotopic links T_0 and T_1 are componentwise homologous, in the sense that the maps

$$
\begin{array}{ccc}
H_2(T_0) & \searrow & \\
 & & H_2(M) \\
H_2(T_1) & \nearrow &
\end{array}
$$

induced by the inclusions have the same image.

Now the first part of Theorem 2.1 may be proved as follows, using the a priori weaker hypothesis that T_0 and T_1 are componentwise homologous. The intersection form on M splits as the orthogonal direct sum of the intersection forms on $M - N_i$ and N_i, where N_i denotes an open tubular neighborhood of T_i in M, $i = 0, 1$. Thus the maps

$$\begin{array}{ccc} H_2(M - N_0) & \searrow & \\ & & H_2(M) \\ H_2(M - N_1) & \nearrow & \end{array}$$

induced by the inclusions must also have the same image, and so $M - N_0$ and $M - N_1$ have isomorphic forms. Since $M/T_i = (M - N_i) \cup$ 4-balls, M/T_0 and M/T_1 also have isomorphic forms. Therefore they are h-cobordant [Wall].

It follows from Theorem 2.1 that the notions of concordance, homotopy, and componentwise homology are the same, up to equivalence, for unit links in closed, simply connected 4-manifolds.

§3. Extending Diffeomorphisms of 3-Manifolds

In the next two sections we explore the problem of extending diffeomorphisms of 3-manifolds to 4-manifolds which they bound. We will make use of the methods developed in this section to prove (via Lemma 5.1) the main results of the paper in §5 and §6. The following section (§4) will not be used in the sequel, but is of independent interest.

Let Q be an oriented 4-manifold with boundary, and let h be an orientation preserving diffeomorphism of ∂Q. We are interested in the following

Question 3.1 Does h extend to an orientation preserving diffeomorphism of Q ?

The answer is well known to be "no" in general, even if h induces the identity on homotopy groups. For example the "twist" diffeomorphism τ of $\partial(S^2 x B^2) = S^2 x S^1$ defined in §1 does not extend to $S^2 x B^2$. This may be seen by observing that the two manifolds

$$S = (S^2 x B^2) \underset{id}{\cup} (S^2 x B^2)$$

$$T = (S^2 x B^2) \underset{\tau}{\cup} (S^2 x B^2)$$

are not diffeomorphic. In fact S and T are just the two 2-sphere bundles over S^2, which are not even homotopy equivalent. It follows that τ does not even extend to a homotopy equivalence of $S^2 x B^2$.

Thus one may attack this problem from a homotopy theoretic point of view, looking for obstructions to extending h to a homotopy equivalence. J. Morgan, for example, has pursued this approach in the case of simply connected Q (unpublished), as have Cappell - Shaneson and Gordon for certain bundles Q over the circle [Cappell-Shaneson] [Gordon].

We take a different tact, as we shall need positive results for certain explicit examples of 4-manifolds Q and diffeomorphisms h of ∂Q. If, for example, all the available obstructions to extending a particular h vanish (or we do not know how to calculate them), then there is some hope that h will extend.

We give a constructive method for how to proceed when Q and h satisfy the following conditions

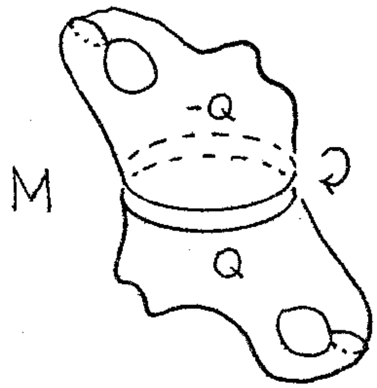
(1) Q is obtained from the 4-ball by adding 2-handles

(2) h is given as the restriction of an explicit diffeomorphism of a 4-manifold P obtained from Q by blowing up points. By "explicit" we mean given as a sequence of handle slides (see Remark 3.3 (5) below for details).

Preliminary Definitions and Discussion

Let Q be an oriented 4-manifold.

By $\Gamma(Q)$ we denote the set of diffeomorphism classes of pairs (M, Q) with $\overline{M - Q} = -Q$. Thus M is formed by identifying Q with $-Q$ by some orientation preserving diffeomorphism of ∂Q.

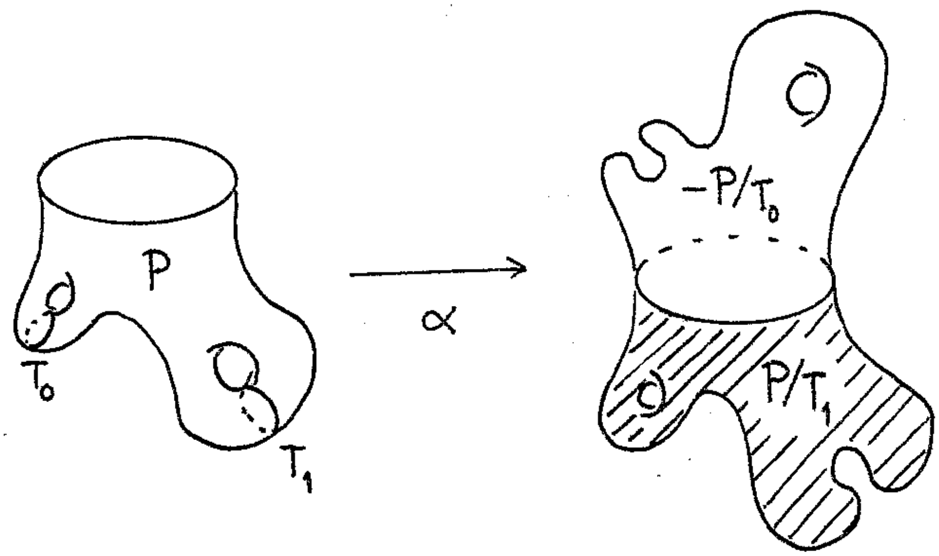


Next consider triples (P, T_0, T_1) where T_0 and T_1 are unit links in an oriented 4-manifold P with P/T_0 and P/T_1 diffeomorphic to Q. Two triples (P, T_0, T_1) and (P', T_0', T_1') are <u>equivalent</u> if there is a diffeomorphism $f: P \longrightarrow P'$ with $f(T_i)$ isotopic to T_i', $i = 0, 1$. Let $\Lambda(Q)$ denote the set of equivalence classes of such triples.

There is a (well defined) map

$$\Lambda(Q) \xrightarrow{\alpha} \Gamma(Q)$$

given by $\alpha(P, T_0, T_1) = ((P \underset{id}{\cup} -P)/(T_1 \cup -T_0), P/T_1)$. In other words, $\alpha(P, T_0, T_1)$ is obtained from the double of P by blowing down T_0 in one copy of P and T_1 in the other.



The relevance of α to the question of extending diffeomorphisms of ∂Q will be explained below.

First consider the group $\mathrm{Diff}(\partial Q)$ of orientation preserving diffeomorphisms of ∂Q. Let

$$\mathrm{Diff}(Q) \xrightarrow{r} \mathrm{Diff}(\partial Q)$$

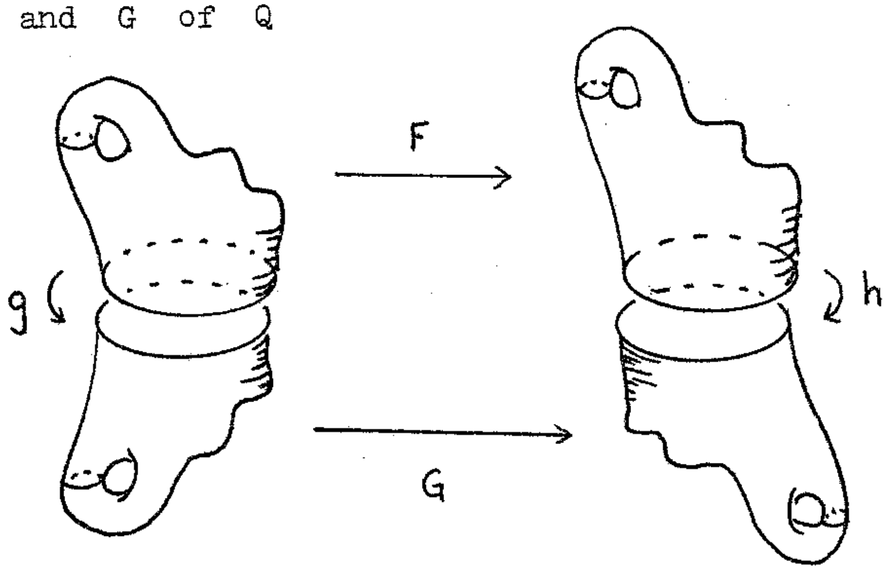
denote the restriction homomorphism. Then we have

<u>Proposition 3.2</u> The double cosets of the image **R** of r in $\mathrm{Diff}(\partial Q)$ are in one to one correspondence with the elements of $\Gamma(Q)$. **i.e. RhR**

<u>Proof</u> Let $[h]$ denote the double coset represented by a diffeomorphism h of Q. Then the map **should be the boundary of Q**

$$[h] \longrightarrow (Q \cup_h -Q,\ Q)$$

sets up the desired correspondence. It is surjective since every element (M, Q) in $\Gamma(Q)$ is diffeomorphic to a pair $(Q \cup_h -Q,\ Q)$, for some h. It is injective because $(Q \underset{g}{\smile} -Q, Q)$ and $(Q \underset{h}{\smile} -Q, Q)$ are pairwise diffeomorphic if and only if there are diffeomorphisms F and G of Q



with $g = (G^{-1}|\partial Q)h(F|\partial Q)$, i.e. $[g] = [h]$.

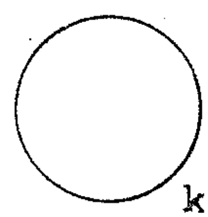
Thus the elements of $\Gamma(Q)$ may be thought of as diffeomorphisms of ∂Q up to composition on either side by diffeomorphisms which extend to Q. In particular, the element $(Q \underset{id}{\smile} -Q, Q)$ corresponds to the diffeomorphisms of ∂Q which extend to Q. We denote it by 1.

Now we may interpret $\alpha : \Lambda(Q) \longrightarrow \Gamma(Q)$ as a restriction map in the following sense. For any element (P, T_0, T_1) of $\Lambda(Q)$ choose a diffeomorphism $h: P \to P$ for which $h(T_1) = T_0$. It is straightforward to verify (along the lines of 3.2) that $\alpha(P, T_0, T_1) = [h|\partial Q]$ for any such h, where ∂Q and ∂P ($= \partial P/T_0$) are identified using any diffeomorphism between Q and P/T_0.

Kirby's Calculus

We now restrict our attention to 4-manifolds Q obtained by adding 2-handles to the 4-ball. Such a manifold may be described by a <u>framed link</u> in S^3, consisting of the attaching circles of the 2-handles together with (integer) framings for their normal bundles. The 4-manifold obtained from a given framed link L will be denoted by M_L.

For example, the framed link



defines the disc bundle over S^2 with Euler class k.

We will assume that the reader is somewhat familiar with this point of view, as developed in [Kirby_1]. We recall the two operations O_1 and O_2 (the Calculus) defined there.

The first operation O_1 changes a framed link L by adding an unknotted circle K with framing ± 1 which lies in a 3-ball disjoint from L. In M_L it corresponds to blowing up a point.

The inverse operation O_1^{-1} removes a component K of L as above and corresponds to blowing down a 2-sphere S in M_L.

In particular, S is just the core of the 2-handle over K together with an unknotted disc in B^4 bounded by K (this characterizes S up to isotopy). We say that K represents S, and denote S by $[K]$. Analogously, if L_0 is a sublink of L whose components are unknotted and mutually unlinked, then L_0 represents a link $[L_0]$ of 2-spheres in M_L consisting of the cores of the 2-handles attached to L_0 together with the obvious collection of discs in B^4 bounded by L_0.

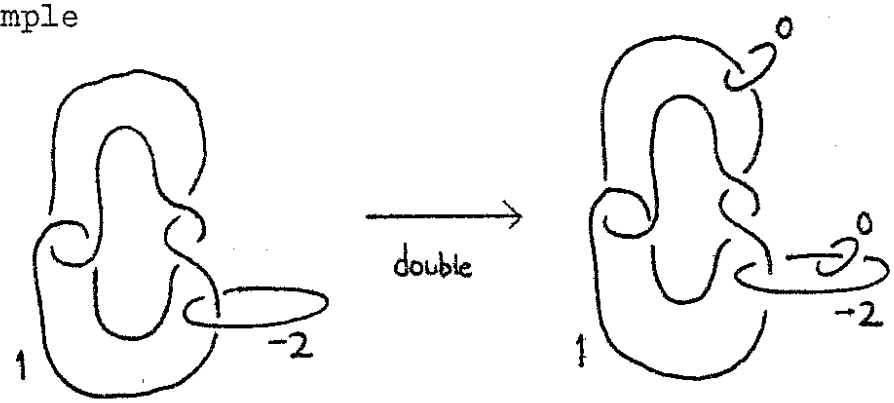
The second operation O_2 replaces some component J of L by J', a band connected sum of J with the push off of some other component K. The framings change accordingly (see [Kirby$_1$] for details). We say that the resulting link L' is obtained by sliding J over K since it corresponds to sliding the associated 2-handles in M_L over each other. Note that M_L and $M_{L'}$ are diffeomorphic.

The theorem in [Kirby$_1$] states that ∂M_L and $\partial M_{L'}$ are diffeomorphic (preserving the natural orientations induced from the orientation on B^4) if and only if there is a sequence of operations $O_1^{\pm 1}$ and O_2 carrying L to L'. We call such a sequence p a path in the Calculus and usually denote L' by $p(L)$.

We will assume that all our paths p are ordered, in the sense that they may be written as a "composition" of paths $p = p_d p_s p_u$ where p_u involves only blowing up (O_1), p_s involves only sliding (O_2), and p_d involves only blowing down (0_1^{-1}). (Hence p is ordered up, slide, down!)

Now we make some important remarks, all of which will be referred to in the sequel.

Remarks 3.3 (1) If $Q = M_L$ for some framed link L, then the double $Q \underset{id}{\smile} -Q$ of Q may be gotten by attaching 2-handles to Q along the boundaries of the cocores of the 2-handles of $Q = M_L$, and then capping off with a 4-handle. These new 2-handles will be called the dual handles to the 2-handles in M_L. It follows easily that, without the 4-handle, the double of Q may be described by the framed link $L \cup L^*$, where L^* is a collection of meridians for the components of L, each with framing zero. For example



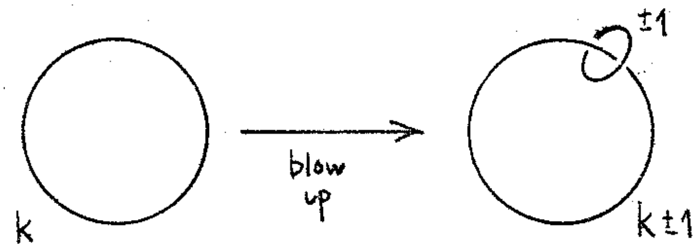
In the sequel L^* will always denote the attaching circles for the dual 2-handles.

Thus we have

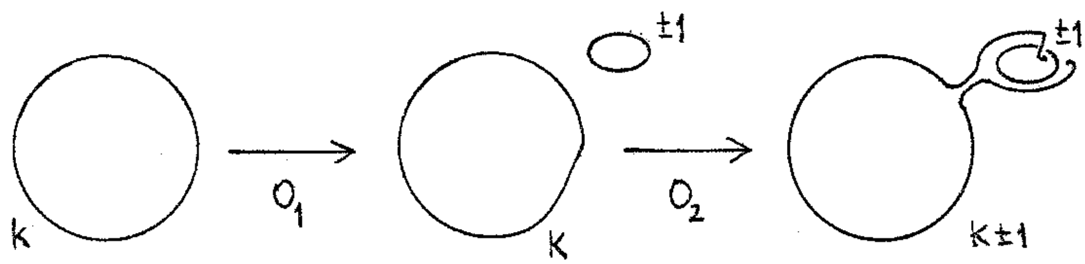
$$M_L \underset{id}{\smile} -M_L \cong \hat{M}_{L \cup L^*}$$

where $\wedge$ denotes capping off.

(2) If we wish to blow up a point on the 2-sphere [K] represented by some (unknotted) component K of a framed link L, then we add to L a meridian of K with framing ± 1



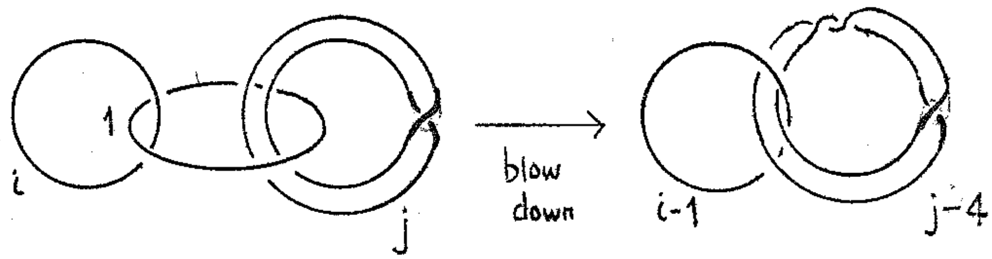
This may be thought of as a composition of operations O_1 and O_2



The image of [K] under this blowing up is the 2-sphere represented by the "same" circle K (whose framing has changed by ± 1).

(3) To blow down the 2-sphere [K] represented by an unknotted component K of L with framing ± 1, we slide over K every component of L which links K, thereby freeing K to be blown down using O_1^{-1}. The reader may verify that this has the effect of giving all the components which link K a full left or right handed twist (changing the framings accordingly) and then

removing K (cf. Propositions 1A and 1B in [Kirby$_1$]).
For example

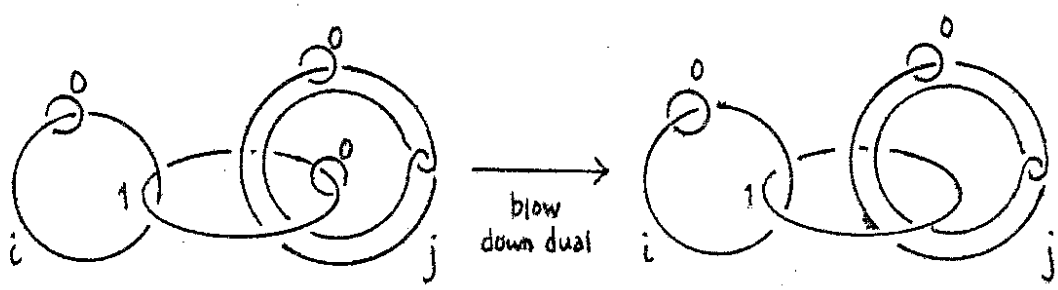


(4) Suppose L is a framed link. The previous remark shows how to blow down the 2-sphere [K] represented by an appropriate component K of L. Consider the "dual" 2-sphere $[K]^*$ in the double $M_L \underset{id}{\cup} -M_L$ of M_L, that is the reflected image of [K] through ∂M_L.

We assert that blowing down $[K]^*$ in the double $\hat{M}_{L \cup L^*}$ of M_L has the effect of removing the dual handle over K^*. That is

$$(\hat{M}_{L \cup L^*}/[K]^*,\ M_L) = (\hat{M}_{L \cup (L^* - K^*)},\ M_L)$$

For example



First observe that we may assume that K is free from the other components of L, as we may slide any component of L over K without touching [K]. That is

$$(M_L, [K]) \cong (M_{p(L)}, [K])$$

for any path p sliding components of L - K over K.

Now for the framed knot K, we have

$$(\hat{M}_{K \cup K^*}/[K^*], M_K) \cong (\hat{M}_K, M_K)$$

as $\hat{M}_K = CP^2$, $\hat{M}_{K\cup K^*} = CP^2 \# -CP^2$, and $[K^*] = -CP^1$. The result follows easily.

(5) As we remarked above, if p is a path in the Calculus (starting at L) which consists only of handle slides ($p = p_s$), then M_L and $M_{p(L)}$ are diffeomorphic. In fact there is a natural diffeomorphism (up to isotopy)

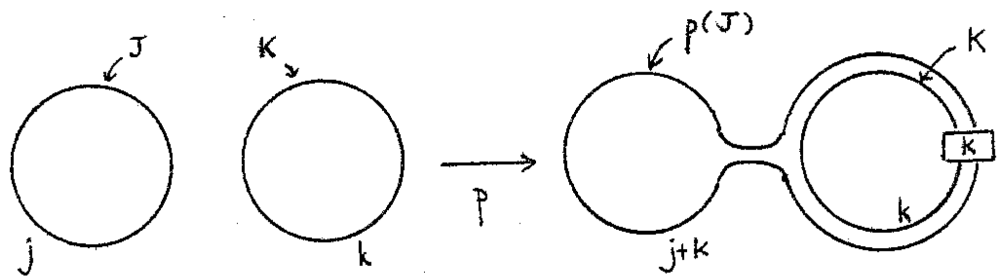
$$h_p : M_L \longrightarrow M_{p(L)}$$

defined as follows. For simplicity we assume that p consists of a single handle slide of a component J of L over some other component K. In general h_p will be a composition of the diffeomorphisms obtained from these "elementary" paths.

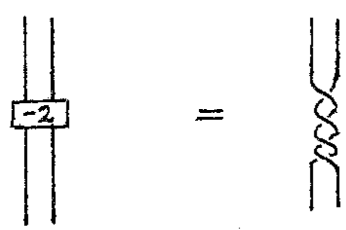
Off of a collar neighborhood U of M_{L-J} in M_{L-J}, let h_p be the identity. On U define h_p to be an isotopy (given by the particular handle slide) carrying J to p(J). Now h_p extends over the 2-handle attached to J.

We call any such diffeomorphism of M_L an <u>explicit</u> diffeomorphism.

(6) Suppose J and K are unknotted and unlinked components of a framed link L and p is the elementary path consisting of a single slide of J over K along the trivial band



Here $\boxed{k}$ denotes k full twists. For example,



Consider the 2-sphere S in M_L obtained by trivially tubing together $[J]$ and $[K]$. Then it is not difficult to verify that

$$S = h_p^{-1}[p(J)]$$

We leave this to the reader.

Extending Diffeomorphisms

Henceforth we fix a framed link L and set $Q = M_L$, $\Lambda_L = \Lambda(M_L)$ and $\Gamma_L = \Gamma(M_L)$.

<u>Definition 3.4</u> By a <u>loop in the Calculus</u> (based at L) we mean an ordered path p in the Calculus with $p(L) = L$ (equality means isotopy). Let Ω_L denote the set of all such loops.

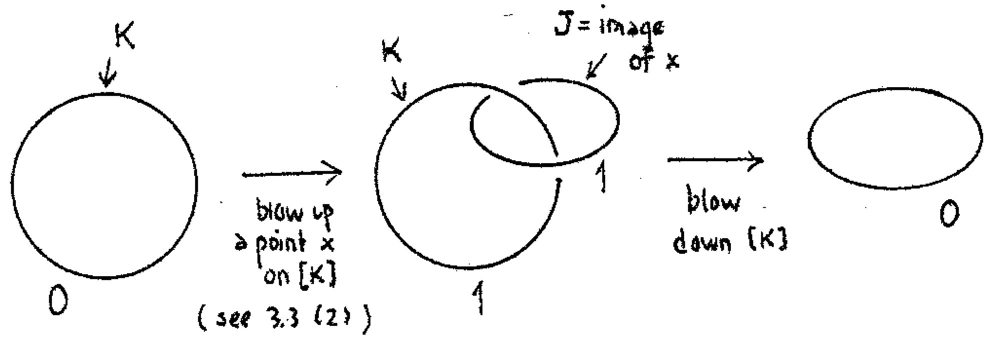
Consider the map

$$\Omega_L \xrightarrow{\beta} \Lambda_L$$

given by

$$\beta(p) = (M_{p_u(L)}, [p_u(L) - L], h_{p_s}^{-1}[p_s p_u(L) - p(L)])$$

For example, if p is the path



then $\beta(p) = (M_{J\cup K}, [J], [K])$.

Now set $\gamma = \alpha\beta$, where $\alpha : \Lambda_L \to \Gamma_L$ is the map defined earlier in this section. We obtain a diagram

$$\begin{tikzcd} \Omega_L \arrow[d, "\beta"'] \arrow[dr, "\gamma"] & \\ \Lambda_L \arrow[r, "\alpha"'] & \Gamma_L \end{tikzcd}$$

For p in Ω_L, the element $\gamma(p)$ may be interpreted (using the remarks following Proposition 3.2) as the restriction to ∂M_L of an explicit diffeomorphism $h: P \to P$ where P is obtained by blowing up points in M_L. In particular, $P = M_{p_u(L)}$ and $h = h_{p_s}^{-1} q$, where q is any natural identification of $M_{p_u(L)}$ with $M_{p_s p_u(L)}$ induced by a diffeomorphism of S^3 carrying $(p_u(L), L)$ to $(p_s p_u(L), p(L))$.

In other words, any loop p in the Calculus (based at L) defines an equivalence class $(\gamma(p))$ of diffeomorphisms of ∂M_L. The question of whether these diffeomorphisms extend to M_L is just the question of whether $\gamma(p) = 1$ in Γ_L.

Observe that the theorem in [Kirby_1] shows that $\beta: \Omega_L \to \Lambda_L$ is surjective. We will see in the next section that $\alpha: \Lambda_L \to \Gamma_L$ is also surjective. It follows that γ is surjective, and so every diffeomorphism of ∂M_L arises as above from a loop in the Calculus.

Thus Question 3.1 for $Q = M_L$ reduces to the problem of identifying the kernel $\gamma^{-1}(1)$ of γ.

We give a sufficient condition for a loop to be in $\ker\gamma$ in Proposition 3.7 below. But first we need a couple of lemmas and a definition.

Lemma 3.5 Suppose that L_0 and L_1 are disjoint framed links, and p is a path in the Calculus starting at $L_0 \cup L_1$ and consisting only of handle slides over components of L_0. Then there is a pairwise diffeomorphism

$$(M_{L_0 \cup L_1}, M_{L_0}) \xrightarrow{h_p} (M_{p(L_0 \cup L_1)}, M_{p(L_0)})$$

Remark If we add a collar to the boundary of the first factor of each pair, then the same result holds allowing components of L_1 to slide over each other.

Proof of 3.5 We may assume that p consists of a single handle slide of some J over K. The general result follows by induction.

There are two possibilities. Either J and K are both in L_0, or J is in L_1 and K is in L_0. In either case, p restricts to an operation on $L_0 \cup J$, and so there is an explicit diffeomorphism (see Remark 3.3 (5))

$$M_{L_0 \cup J} \xrightarrow{h_p} M_{p(L_0 \cup J)}$$

which may be chosen to be the identity on $L_1 - J$. Thus h_p extends over the handles attached to $L_1 - J$. Clearly $h_p(M_{L_0} - J) = M_{p(L_0 - J)}$ and $h_p(M_{L_0 \cup J}) = M_{L_0 \cup J}$, so in both cases above we have $h_p M_{L_0} = M_{p(L_0)}$.

Suppose L is a sublink of some framed link L', and p is in Ω_L. Let $p(L')$ denote any framed link obtained by "carrying along" the components of $L' - L$ while performing the operations of p. There is a choice involved whenever we slide handles, as the bands along which we slide may link $L' - L$ arbitrarily. Consequently $p(L')$ is not uniquely defined. However L is always a sublink $(p(L))$ of $p(L')$, and it is not difficult to show that the pair $(M_{p(L')}, M_L)$ is well defined up to diffeomorphism.

In fact, each choice for $p(L')$ corresponds to an (ordered) path in the Calculus from L' to $p(L')$. For any such path p', the previous lemma provides a diffeomorphism $h = h_{p'_s}$

$$(M_{p'_u(L')}, M_{p_u(L)}) \xrightarrow{h} (M_{p'_s p'_u(L')}, M_{p_s p_u(L)})$$

It follows from the proof of the lemma that for any other diffeomorphism $g = h_{q'_s}$ arising from another such path q', we have

$$hg^{-1}(T) = T$$

where T is the collection of 2-spheres $[p_s p_u(L) - p(L)]$ in $M_{p_s p_u(L)}$ to be blown down. Thus hg^{-1} induces a pairwise diffeomorphism between $(M_{p'(L')}, M_L)$ and $(M_{q'(L')}, M_L)$.

Now specializing to the case $L' = L \cup L^*$, we see that $(M_{p(L\cup L^*)}, M_L)$ is well defined, up to diffeomorphism, for any p in Ω_L. Therefore $(\hat{M}_{p(L\cup L^*)}, M_L)$ defines an element of Γ_L. This gives a useful form for the map $\gamma : \Omega_L \to \Gamma_L$.

<u>Lemma 3.6</u> $\gamma(p) = (\hat{M}_{p(L\cup L^*)}, M_L)$

<u>Proof</u> Recall $\gamma = \alpha\beta$, and so setting $T_0 = [p_u(L) - L]$ and $T_1 = h_{p_s}^{-1}[p_s p_u(L) - p(L)]$, we have

$$(p) = ((M_{p_u(L)} \underset{id}{\smile} -M_{p_u(L)})/(T_1 \cup -T_0),\ M_{p_u(L)}/T_1)$$

$$= (\hat{M}_{p_u(L)\cup p_u(L)^*}/(T_1 \cup -T_0),\ M_{p_u(L)}/T_1)$$

which by Remark 3.3 (4)

$$= (\hat{M}_{p_u(L)\cup(p_u(L)^* - L^*)}/T_1,\ M_{p_u(L)}/T_1)$$

$$= (\hat{M}_{p_u(L\cup L^*)}/T_1,\ M_{p_u(L)}/T_1)$$

$$= (\hat{M}_{p_s p_u(L\cup L^*)}/h_{p_s}(T_1),\ M_{p_s p_u(L)}/h_{p_s}(T_1))$$

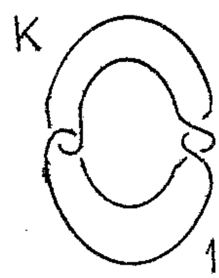
which by Remark 3.3 (3)

$$= (\hat{M}_{p(L\cup L^*)},\ M_L).$$

Combining 3.5 and 3.6, we have

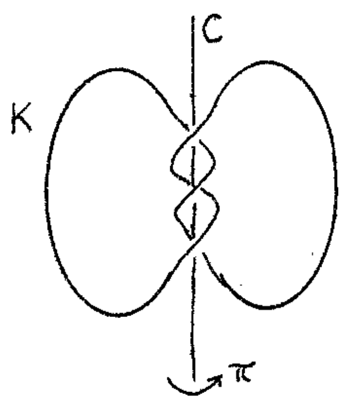
<u>Proposition 3.7</u> If p is in Ω_L and $L \smile L^*$ can be obtained from $p(L \smile L^*)$ by sliding components of $p(L \smile L^*)$ over components of $p(L) = L$, then $\gamma(p) = 1$.

<u>Example 3.8</u> let $Q = M_K$, where K is the right handed trefoil with framing 1



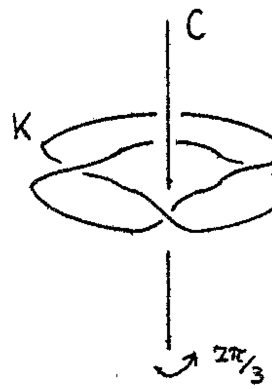
Then ∂Q is the Poincaré homology 3-sphere. We show that there are diffeomorphisms of ∂Q of orders 2, 3, and 5 which extend to diffeomorphisms of Q.

The ones of order 2 and 3 are easy to construct (without using 3.7). For example, to obtain one of order 2 we observe that the trefoil has a 2-fold symmetry of rotation about an unknotted circle C in S^3



It follows that there is an orientation preserving involution on B^4 mapping K to itself. This clearly extends over the 2-handle attached to K, yielding an involution on Q. The restriction of this involution to ∂Q is the desired diffeomorphism of ∂Q of order 2.

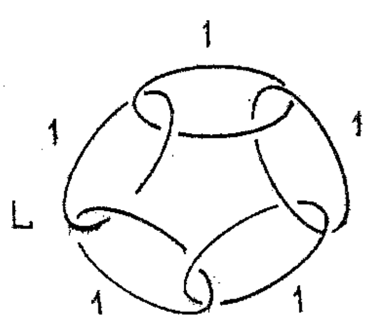
A similar argument, exploiting the 3-fold symmetry of the trefoil



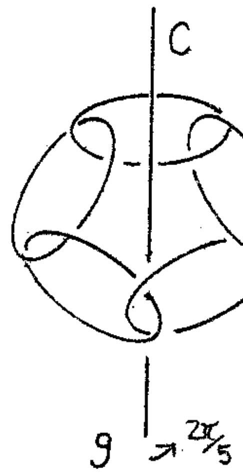
provides a diffeomorphism of ∂Q of order 3 which extends to Q.

Now consider ∂Q as the 5-fold cyclic branched cover of the trefoil (see for example [Kirby-Scharlemann]). Any covering translation h of ∂Q provides a diffeomorphism of order 5. We describe one such h explicitly below.

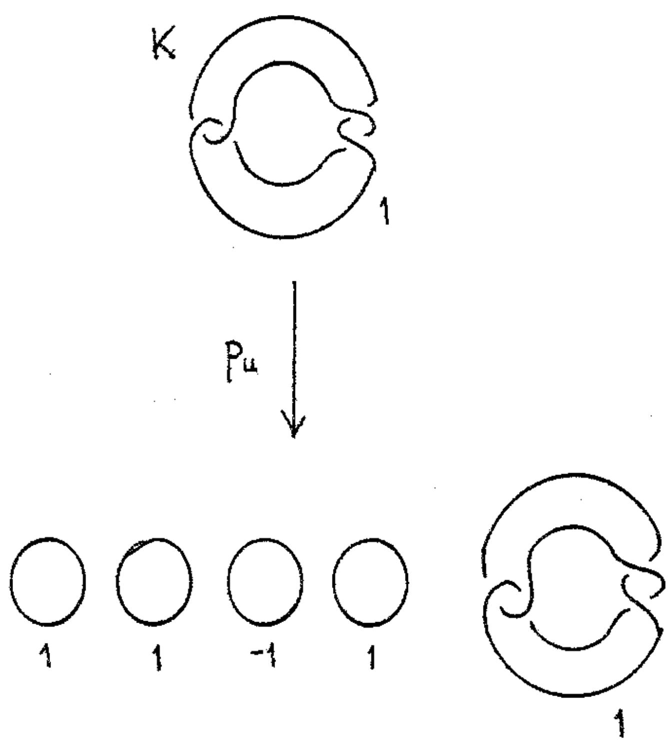
We may view the 5-fold cover of the trefoil as the boundary of the 4-manifold P given by the following framed link L of five circles

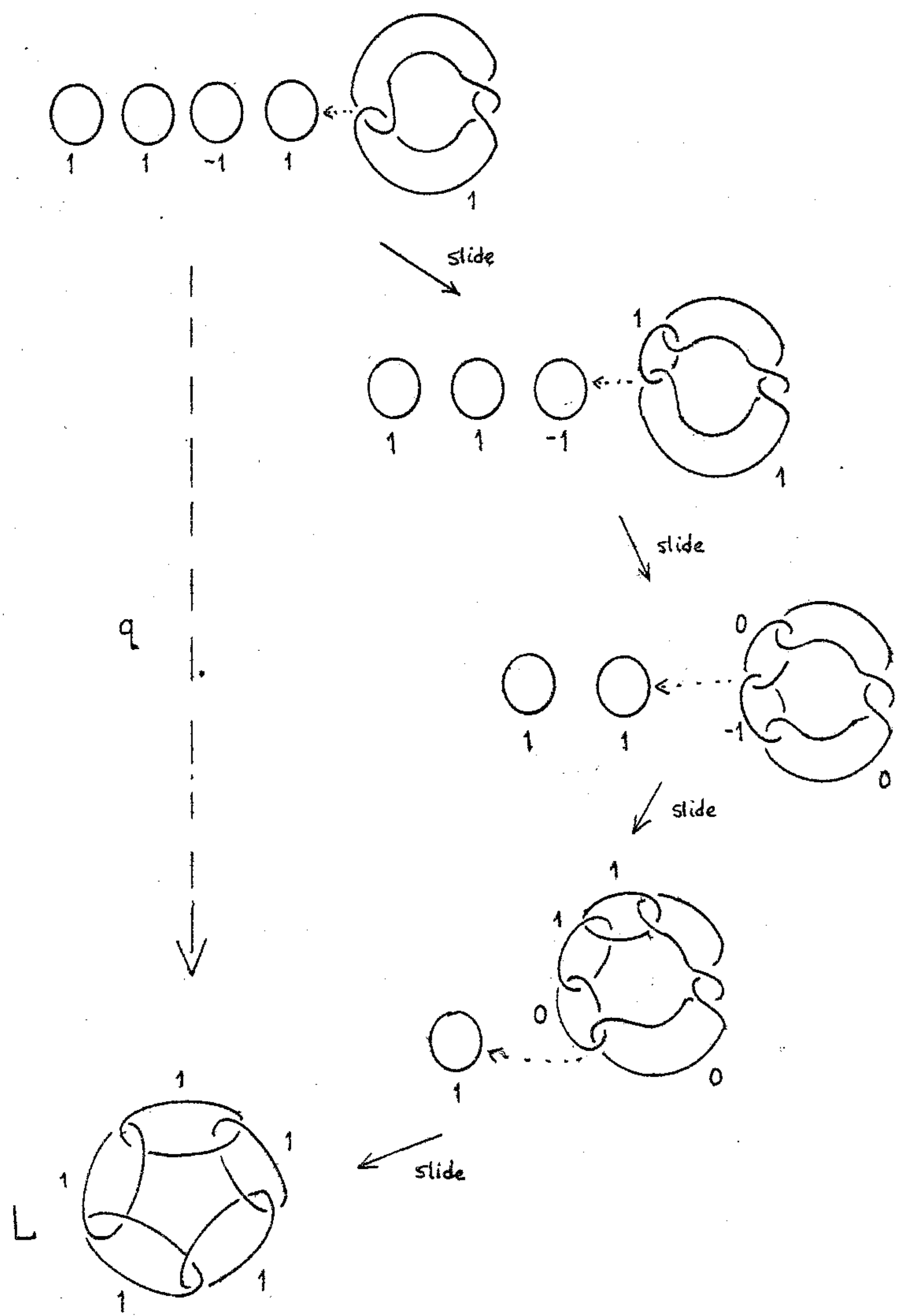


A generator for the covering translations on ∂M_L may be given as the restriction of the obvious diffeomorphism g of M_L of order 5 obtained (as in the cases above) from the 5-fold symmetry of L about an unknotted circle C in S^3



Now there is a path qp_u in the Calculus from K to L as follows




1
1
-1
1
1
slide
1
1
1
-1
1
slide
0
1
1
-1
0
slide
1
1
0
1
0
q
1
1
1
L
1
1
slide

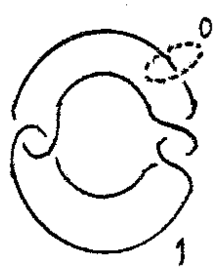
If q' denotes the "inverse" path to q acting on $g(L)$, then the composed path $p_s = q'q$ is an element of $\Omega p_u(K)$. Observe that the diffeomorphism

$$h_{p_s}: M_{p_u}(K) \longrightarrow M_{p_u}(K)$$

is just $h_q^{-1} g h_q$, and so its restriction h to $\partial M_{p_u}(K) = \partial Q$ is periodic of order 5.

To verify that h extends to Q, it suffices (by Proposition 3.7) to check that the loop $p = p_d p_s p_u \in \Omega_K$ (where p_d is the obvious blowing down) is in $\ker\gamma$ for $\gamma: \Omega_K \to \Gamma_K$ as defined above.

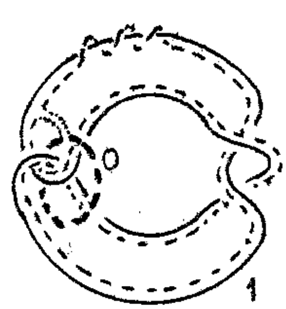
We calculate $K \cup K^*$ and $P(K \cup K^*)$ to be



$K \cup K^*$

$p(K \cup K^*)$

where the dual circles are dotted. Sliding the dotted circle in $p(K \cup K^*)$ over the trefoil once we get



which may be isotoped to

Thus $p \in \ker\gamma$ and h extends.

§4. A Digression

In this section we prove the following theorem.

Theorem 4.1 Let $h: N \to N$ be an orientation preserving diffeomorphism of a closed, orientable 3-manifold.

Then for some simply connected 4-manifold P, there is an orientation preserving diffeomorphism $H: P \longrightarrow P$ and a diffeomorphism $i: N \longrightarrow P$ for which

$$H|\partial P = ihi^{-1}$$

Remark This extends the work of M. Kreck on oriented bordism of diffeomorphisms of odd dimensional manifolds. We recall that two pairs (N_i^n, h_i), where h_i is a diffeomorphism of N_i, are bordant if there is an $(n+1)$-manifold V and a diffeomorphism H of V such that $\partial V = N_0 \cup -N_1$ and $H|N_i = h_i$. Bordism classes of diffeomorphisms of n-manifolds form an abelian group Δ_n under disjoint union. The main result of [Kreck] is that

$$\Delta_n = \Omega_n \oplus \hat{\Omega}_{n+1}$$

for n odd and $\neq 3$. Here Ω_* denotes oriented bordism of manifolds and $\hat{\Omega}_*$ denotes the kernel of the signature homomorphism $\Omega_* \to Z$. Theorem 4.1 shows that $\Delta_3 = 0$, which removes the restriction $n \neq 3$ above.

Before giving the proof of 4.1 we need a lemma. Recall the map $\Lambda(Q) \xrightarrow{\alpha} \Gamma(Q)$ defined in the last section.

Lemma 4.2 If Q is a compact, simply connected 4-manifold, then

$$\Lambda(Q) \xrightarrow{\alpha} \Gamma(Q)$$

is surjective.

Proof Without loss of generality, we may assume that the intersection form on Q is odd, for it is evident that the lemma must hold for Q if it holds for $Q \# CP^2$.

Let (M, Q) be an element of $\Gamma(Q)$. By Novikov additivity the signature of M is zero, and so M bounds a 5-manifold W [Rohlin]. We may assume (after surgering W if necessary) that there is a Morse function

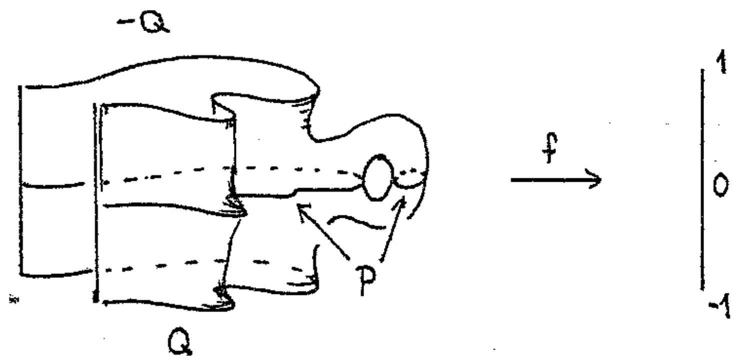
$$f: W \longrightarrow [-1, 1]$$

satisfying

(1) $f^{-1}(-1, 1) \cap \partial W$ is an open tubular neighborhood $\partial Q x(-1, 1)$ of $\partial Q = \partial Q x 0$ with $f^{-1}(t) = \partial Q x t$

(2) Every critical point of f is of index 2 or 3, with values less or greater than zero, respectively

Let P denote the 4-manifold $f^{-1}(0)$



Consider the 5-manifolds $W_t = f^{-1}[-t, t]$ for $t \in (0, 1]$. For t small, $W_t = P \times I$, and so $\partial W_t = P \underset{id}{\smile} -P$. As $\pm t$ crosses a critical value of f, a 3-handle is added to W_t with attaching map in P or $-P$.

The effect on ∂W_t of adding this 3-handle is to blow down a pair of unit knots in P or $-P$. For, inverting the picture, it suffices to show that the effect on the boundary of a 5-manifold of adding a 2-handle is to blow up a pair of points, provided the boundary is simply connected and has an odd intersection form (the ∂W_t are odd since Q is). The first condition shows that adding a 2-handle results in taking the connected sum on the boundary with a 2-sphere bundle T over S^2. The second condition shows that we may choose T to be the non-trivial bundle. But then $T = CP^2 \# -CP^2$, and so the net effect is to blow up a pair of points.

Continuing in this way we obtain unit links T_0 and T_1 in P for which

$$(P \underset{id}{\smile} -P)/(T_1 \cup -T_0) = \partial W_1 = M$$

Since T_0 and T_1 lie away from ∂P, this diffeomorphism identifies P/T_1 with Q.

Proof of Theorem 4.1 Choose any compact, simply connected 4-manifold Q with $\partial Q = N$. By Lemma 4.2, if we blow up sufficiently many points in Q we obtain a (simply connected) 4-manifold P with unit links T_0 and T_1 for which $((P \underset{id}{\cup} -P)/(T_1 \cup -T_0), P/T_1)$ and $(Q \underset{h}{\cup} -Q, Q)$ are pairwise diffeomorphic. In particular P/T_0 and P/T_1 are diffeomorphic to Q, so there is a diffeomorphism G of P carrying T_1 to T_0.

We now have the following diagram of pairwise diffeomorphisms

$$\begin{array}{ccc}
((P \underset{id}{\cup} -P)/(T_1 \cup -T_0), P/T_1) & \longrightarrow & (Q \underset{h}{\cup} -Q, Q) \\
\Big\downarrow{\scriptstyle id \cup G^{-1}} & & \\
((P \underset{G|\partial P}{\cup} -P)/(T_1 \cup -T_1), P/T_1) & \longrightarrow & (P/T_1 \underset{g}{\cup} -P/T_1, P/T_1)
\end{array}$$

where g denotes the canonical diffeomorphism of $\partial(P/T_1)$ induced by $G|\partial P$.

Therefore there are diffeomorphisms

$$Q \underset{f}{\overset{e}{\rightleftarrows}} P/T_1$$

for which $g|\partial(P/T_1) = (f^{-1}|\partial Q)h(e^{-1}|\partial(P/T_1))$. It follows that $(efg)|\partial(P/T_1) = (e|\partial Q)h(e^{-1}|\partial(P/T_1))$.

Now ef naturally induces a diffeomorphism F of P. Setting $i = e|N$ (recall that $N = \partial Q$) and $H = FG$

we have

$$H|\partial P = ihi^{-1}$$

as desired.

§5. 0-Concordance

Let $f: T \times I \to M \times I$ be a generic concordance between two links T_0 and T_1 of 2-spheres in a 4-manifold M (see §1), and let

$$\begin{array}{ccc} M \times I & \xrightarrow{q} & I \\ {\scriptstyle p}\downarrow & & \\ M & & \end{array}$$

be the projections. For every regular value t of qf, set

$$T_t = (pf)(qf)^{-1}(t).$$

T_t is an orientable 2-manifold in M. We may arrange that the critical points of qf have distinct values and that for any two critical points x and y

$$\text{index}(x) < \text{index}(y) \Rightarrow qf(x) < qf(y).$$

Then for any critical value s corresponding to a critical point of index j, and for $\varepsilon > 0$ sufficiently small, $T_{s+\varepsilon}$ is obtained (up to isotopy) by adding an embedded j-handle to $T_{s-\varepsilon}$ in M. Such a concordance will be called <u>nice</u>.

If f is a 0-concordance, then each T_t is a link of 2-spheres in M. The basic idea of the proof of Theorem 5.2 below is to blow down each regular level of the concordance and to show that the resulting 4-manifolds do not change as we cross the critical levels. The only difficulty is that T_t will generally not be a unit link, and so we do not know how to blow it down.

We may, however, generalize the notion of blowing down to arbitrary links T as follows. Roughly speaking, we blow up as few points as possible on T to give each component self intersection ± 1, and then blow down the resulting (unit) link.

Precisely, if T consists of only one 2-sphere S with self intersection k, let (M', S') denote the pairwise connected sum

$$(M', S') = (M, S) \# r(\pm(CP^2, P))$$

where $r = \big| |k| - 1 \big|$, P is a projective line cutting $\pm CP^1$ in one point (see §1), and the sign is chosen to agree with the sign of k. For $k = 0$ we choose the positive orientation.

If T has more than one component, we iterate the process above to obtain (M', T') with T' a unit link. We call T' the <u>image</u> of T. Now define

$$M/T = M'/T'$$

Note that it follows from the proof of Proposition 6.2 that the opposite choice of orientation in defining (M', S')

for the case $k = 0$ does not change M/S, essentially because $\pi_1 SO(3) = Z/2Z$.

Now we come to the chief ingredient in the proofs of Theorem 5.2 and Theorem 6.3 in the next section.

<u>Lemma 5.1</u> Let s be a critical value of a nice 0-concordance $f: TxI \to MxI$.

Then $M/T_{s-\varepsilon}$ and $M/T_{s+\varepsilon}$ are diffeomorphic for sufficiently small ε.

<u>Proof</u> By duality we may assume that the index j of the critical point with value s is 0 or 1.

Choose ε small enough so that s is the only critical value in the interval $J = [s-\varepsilon, s+\varepsilon]$. Let

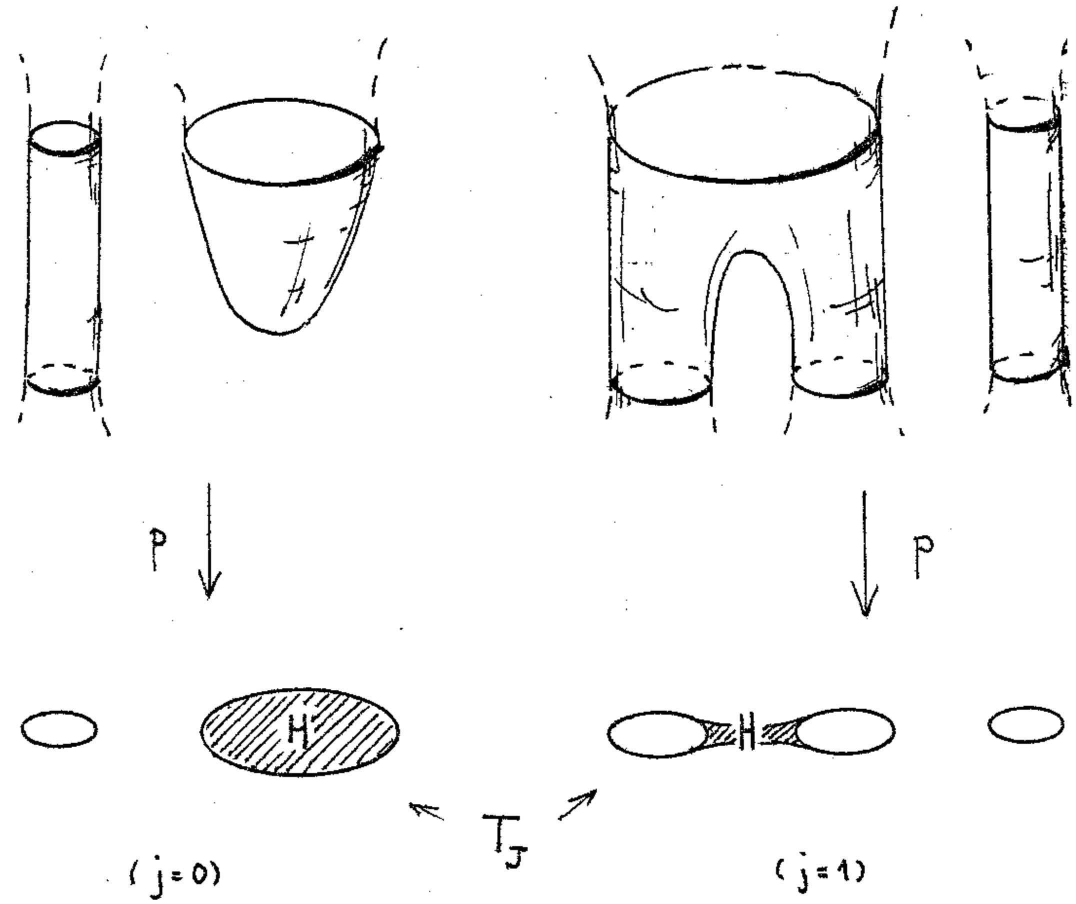
$$T_J = (pf)(qf)^{-1}(J)$$

where p and q denote the projections $M \xleftarrow{p} MxI \xrightarrow{q} I$. We may adjust f by a level preserving isotopy so that $T_J = T_{s-\varepsilon} \cup H$, where $H = B^j x B^{3-j}$ is an embedded j-handle ($j = 0$ or 1) with

$$T_{s-\varepsilon} \cap H = \partial B^j x B^{3-j}$$

$$T_{s+\varepsilon} = (T_{s-\varepsilon} - \partial B^j x B^{3-j}) \cup B^j x \partial B^{3-j}$$

The isotopy class of each T_t remains unchanged, and so M/T_t remains unchanged.



The component of any regular neighborhood of T_J in M which contains the handle H is a 4-manifold P. It is evident that the links $T_{s\pm\varepsilon}$ coincide outside P. We let $T = T_{s\pm\varepsilon} - P$ denote this common link, and set $T_0 = T_{s-\varepsilon} - T$ and $T_1 = T_{s+\varepsilon} - T$.

The rough idea now is that $M/T_{s+\varepsilon}$ is obtained from $M/T_{s-\varepsilon}$ by removing P/T_0 and replacing it with P/T_1. We will see that P/T_0 and P/T_1 are diffeomorphic, and so the problem of showing that $M/T_{s-\varepsilon}$ and $M/T_{s+\varepsilon}$ are diffeomorphic reduces to showing that a particular diffeomorphism of $\partial(P/T_0)$ extends.

The outline of the rest of the proof is as follows.

We first blow up appropriate points in $P \subset M$ to obtain $P' \subset M'$ and unit links T_0' and T_1' in P' for which $M' - P' = M - P$ and

$$M/T_{s-\varepsilon} = M'/(T \cup T_0')$$

$$M/T_{s+\varepsilon} = M'/(T \cup T_1')$$

We then show that there is a diffeomorphism h of P' with $h(T_0') = T_1'$ and $h|\partial P' = \text{identity}$.

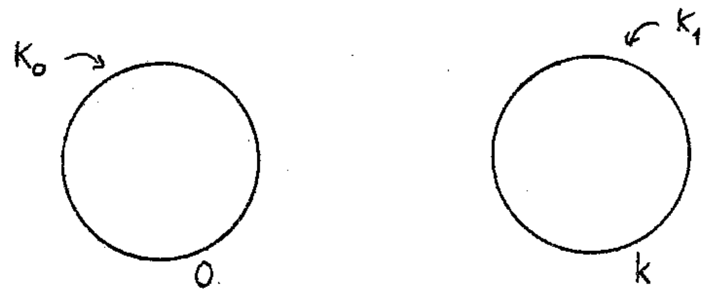
Assuming this, the lemma follows easily. Extending h over $M' - P'$ by the identity, we obtain a diffeomorphism of M' carrying $T \cup T_0'$ to $T \cup T_1'$. The equations displayed above then yield a diffeomorphism between $M/T_{s-\varepsilon}$ and $M/T_{s+\varepsilon}$.

So we must construct T_0', $T_1' \subset P'$ and h as above. We consider the two cases $j = 0$ or 1.

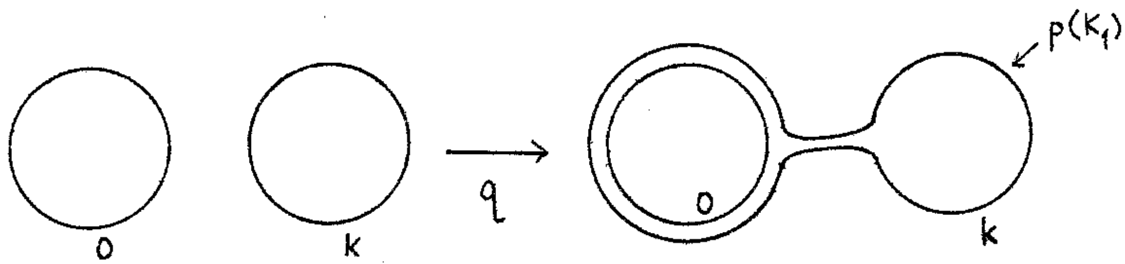
If $j = 0$, then P is a 4-ball with T_0 empty and T_1 an unknotted 2-sphere S inside P. Clearly S must have self intersection zero. Blowing up one point x on S we obtain P', a projective plane CP^2 with a 4-ball B removed. Setting T_0' = image of x and T_1' = image of S, we easily see that the properties above are satisfied. Now T_0' and T_1' are simply a pair of projective lines, and so there is a linear isomorphism of CP^2 carrying one to the other. Adjusting by an isotopy so as to map B to itself by the identity, this restricts to the desired diffeomorphism h of P'.

If $j = 1$, then H joins the two components S_0

and S_1 of T_0, one of which (S_0) must have self intersection zero since f is a concordance. If k is the self intersection of the other component (S_1), then P is diffeomorphic to the boundary connected sum of $S^2 x B^2$ and the disc bundle over S^2 with Euler class k. In other words, P may be described by the framed link of two unknotted circles $K_0 \cup K_1$

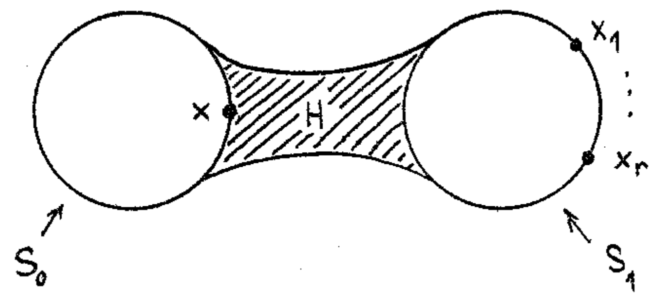


with $S_i = [K_i]$ represented by K_i (see §3). Now T_1 consists of a single 2-sphere S (with self intersection k) obtained by trivially tubing together S_0 and S_1, and so S is isotopic to $h_q^{-1}[p(K_1)]$ where q is the handle slide

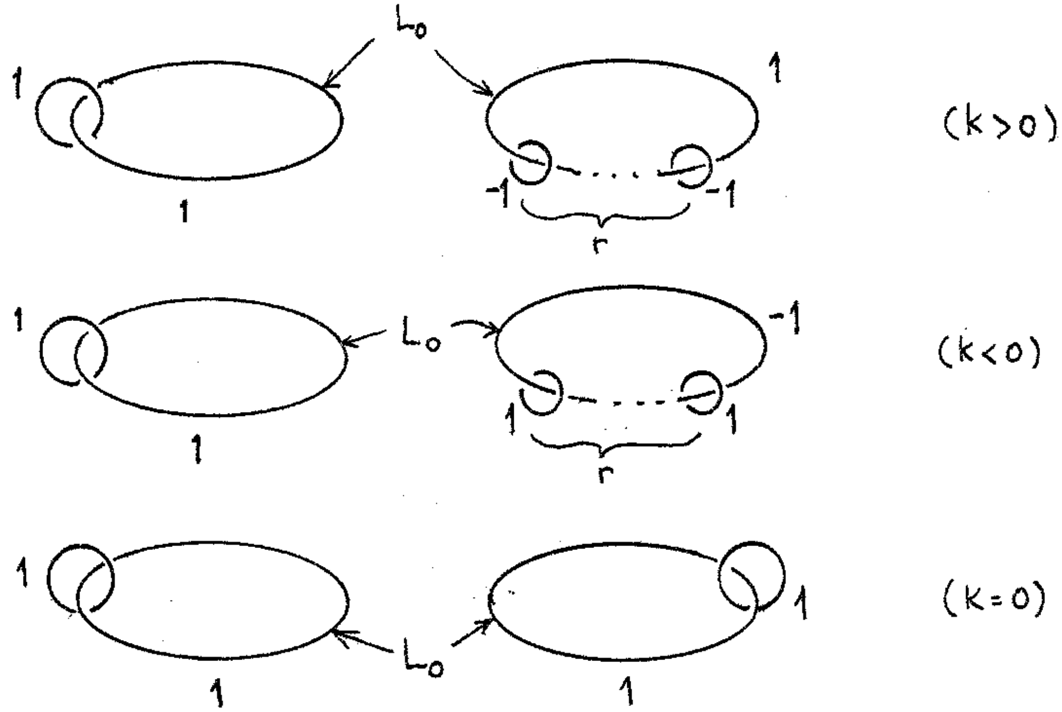


(see §3.3 (6)).

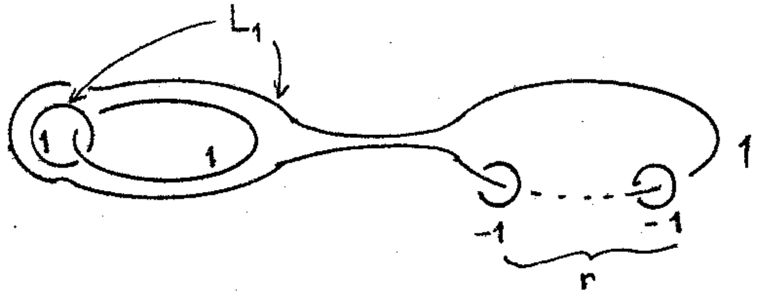
Blowing up $r = \big||k| - 1\big|$ points $x_1, \cdot \cdot \cdot , x_r$ on $S_1 \cap S$ and one point x on $S_0 - S$, as indicated below



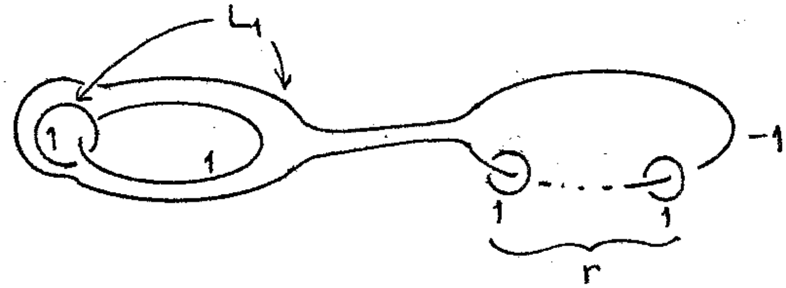
we obtain P', which may be described by the framed link L' (depending on k)



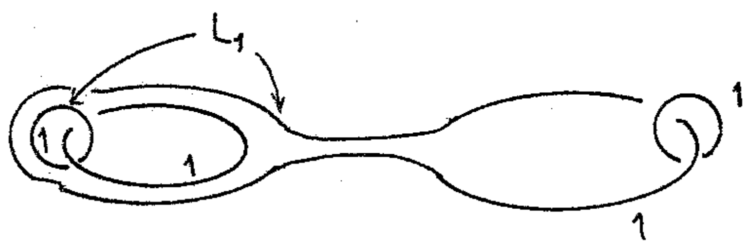
Set T_0' = image of T_0 and T_1' = image of $T_1(=S) \cup$ image of x. Then we have T_0' represented by the link L_0 indicated above, and T_1' represented by the link L_1 indicated in the following link description of P' (arising from $M_{q(K_0 \cup K_1)}$ instead of $M_{K_0 \cup K_1}$)



(k>0)



(k<0)



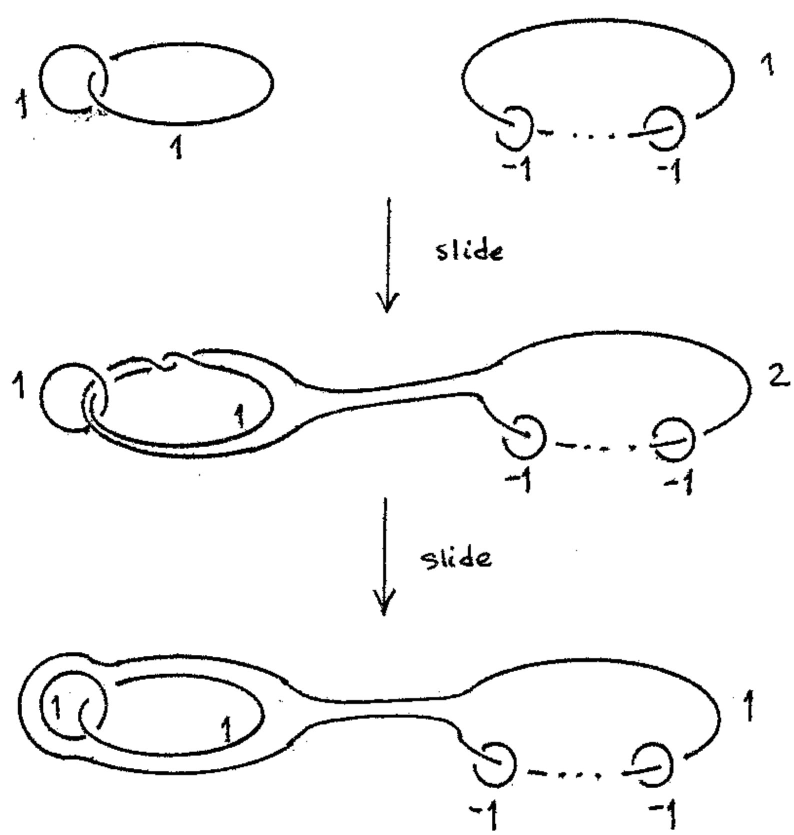
(k=0)

It is clear that $T_0', T_1' \subset P'$ satisfy the desired properties. To construct $h: P' \to P'$ with $h(T_0') = T_1'$ and $h|\partial P' =$ identity it suffices to show that

$$(P', T_0', T_1') \in \ker \alpha$$

where $\alpha: \Lambda(P'/T_0') \to \Gamma(P'/T_0')$ is the map constructed in §3. For $\alpha(P', T_0', T_1') = 1$ yields a pairwise diffeomorphism between $((P' \underset{\mathrm{id}}{\cup} -P')/(T_1' \cup -T_0'), P'/T_1')$ and $((P' \underset{\mathrm{id}}{\cup} -P')/(T_0' \cup -T_0'), P'/T_0')$. Since $\Gamma_4 = 0$, this induces a diffeomorphism of pairs $(P' \underset{\mathrm{id}}{\cup} -P', T_1' \cup -T_0') \to (P' \underset{\mathrm{id}}{\cup} -P', T_0' \cup -T_0')$ which carries P' to itself. In other words, there are diffeomorphisms f and g of P' with $f(T_1') = T_0'$, $g(T_0') = T_0'$, and $f = g$ on $\partial P'$. Then $h = f^{-1}g$ is the desired diffeomorphism of P'.

We observed above that $P' = M_{L'}$, and $T_0' = [L_0]$. The second link description above for P' uses the same link L', and is obtained from the first by the following loop p_s in the Calculus (based at L'). We illustrate the case $k>0$. The other two cases are completely analogous.



Clearly $T_1' = h_{p_s}^{-1}[L_1]$, for L_1 as above.

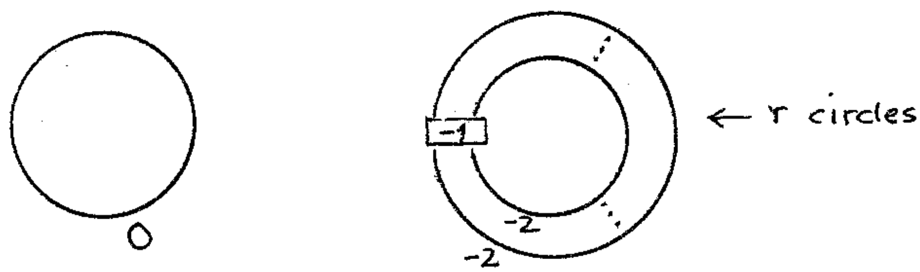
Let p_d be the path in the Calculus which blows down L_0. Set $L = p_d(L_0)$, so that $M_L = P'/T_0'$. If p_u denotes the "inverse" path from L to L', then we obtain a loop $p = p_d p_s p_u$ in the Calculus based at L, i.e. $p \in \Omega_L$. Since $p_u(L) - L = L_0$, it is evident that $\beta(p) = (P', T_0', T_1')$, where $\beta: \Omega_L \to \Lambda_L$ is the map defined in §3.

We now apply Proposition 3.7 to show

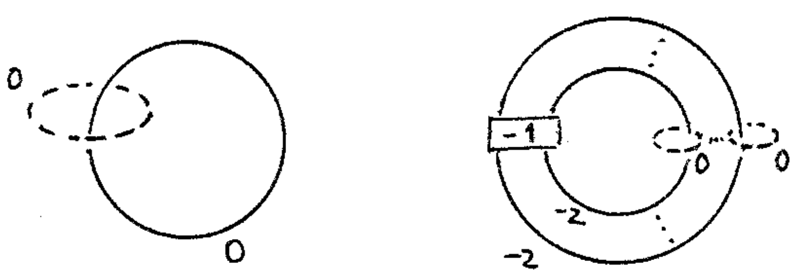
$$p \in \ker \gamma$$

where $\gamma = \alpha\beta$. This gives $(P', T_0', T_1') \in \ker\beta$, and the lemma follows.

Explicitly, we start with L (once again we only carry out the case $k > 0$; the others are analogous)

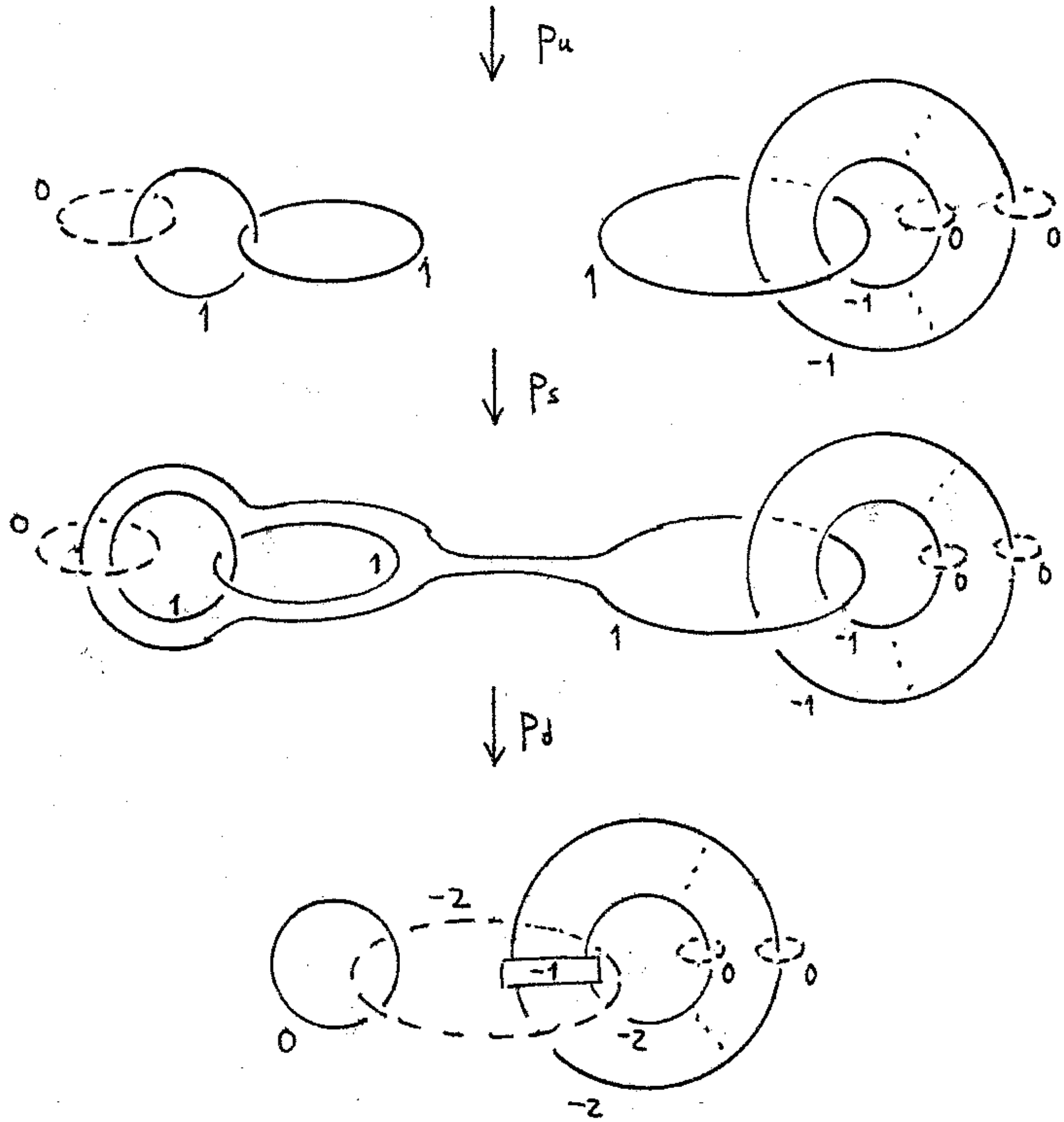


Now $L \cup L^*$ is given by

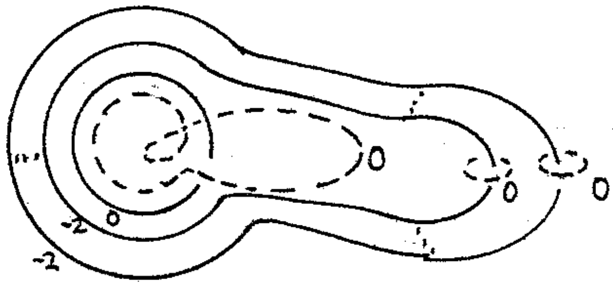


where the dual circles are dotted.

Next we construct $P(L \cup L^*)$ as the final picture in the following sequence



Now $L \cup L^*$ is obtained from $p(L \cup L^*)$ as specified in 3.7 by sliding all the circles with framing -2 (including the dual circle) over the (undotted) circle with framing zero



Thus $p \in \ker \gamma$, as desired.

Inductive application of this lemma to a 0-concordance of unit links yields

Theorem 5.2 0-concordant unit links in a 4-manifold are equivalent.

§6. Embedding CP^1 in CP^2 and Gluck's Construction

In this section we discuss the following conjecture (which was the starting point of our investigations).

<u>Conjecture 6.1</u> Let $f: CP^1 \longrightarrow CP^2$ be a degree one embedding. Then there is a diffeomorphism $h: CP^2 \longrightarrow CP^2$ with $hf = \cancel{f}$. inclusion.

This is merely a restatement of Conjecture 2 in §1, that every unit knot in CP^2 is equivalent to CP^1.

We may reformulate this conjecture in terms of equivariant knot theory of 3-spheres in S^5. Recall that S^5, viewed as the unit sphere in C^3, has a natural S^1 action induced by unit complex multiplication. CP^2 may be defined as the quotient of S^5 by this action. In fact S^5 is a principal $SO(2)$ bundle over CP^2, with the orbits of the action as fibers. The pull back of this bundle under any embedding $f: CP^1 \rightarrow CP^2$ has Euler number equal to the degree of the embedding.* Thus degree one embeddings

---

* The only degrees d realized by embeddings are $|d| \leq 2$ [Tristram]

pull back the Hopf bundle, inducing an equivariant embedding of the 3-sphere in S^5

$$\begin{tikzcd} f^*(S^5) = S^3 \arrow[r] \arrow[d] & S^5 \arrow[d] \\ CP^1 \arrow[r, "f"'] & CP^2 \end{tikzcd}$$

Now any diffeomorphism of CP^2 lifts to an equivariant diffeomorphism of S^5, and conversely any equivariant diffeomorphism of S^5 projects to a diffeomorphism of CP^2. Therefore Conjecture 6.1 is equivalent to the assertion that there is (up to equivariant diffeomorphism of S^5) only one equivariant embedding of S^3 in S^5.

Unfortunately, the conjecture is probably less tractable in this form, as most equivariant problems are studied by factoring out by the group action, which brings us back to where we started.

We observe that if 6.1 fails, then there is a homotopy 4-sphere Σ ($\neq S^4$) for which

$$CP^2 \# \Sigma = CP^2$$

In particular, Σ is obtained by blowing down $f(CP^1)$ in CP^2. Thus 6.1 would follow from the irreducibility of CP^2.

More generally, the equality $CP^2 \# \Sigma = CP^2$ would give $CP^2 \# k\Sigma = CP^2$ for any $k \geq 0$. Thus 6.1 would follow merely from the existence of a bound on the number of factors possible in a connected sum decomposition of CP^2. (Every compact <u>3-manifold</u> has such a bound [Kneser]).

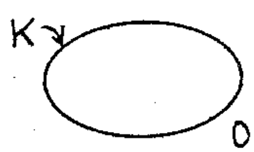
We may study the homotopy 4-sphere $\Sigma = CP^2/f(CP^1)$ from a somewhat different point of view, under the additional assumption that $f(CP^1)$ and CP^1 meet in exactly one point. We do not know if this can always be arranged.

To conform with the notation of §1 we set $S' = f(CP^1)$. Then the image of S' when blowing down CP^1 in CP^2 is a 2-sphere S in S^4. (In other words S' is gotten by blowing up a point of S.) Recalling from §1 H. Gluck's construction of a homotopy 4-sphere τS from a 2-sphere S in S^4, we have

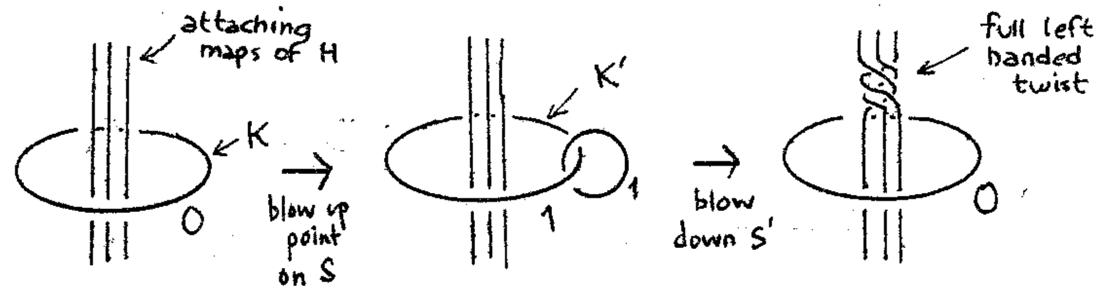
Proposition 6.2 $CP^2/S' = \tau S$

Remark In the notation of §5 this says $S^4/S = \tau S$

Proof of 6.2 View S^4 as a handlebody H built on a tubular neighborhood $S^2 x B^2$ of $S = S^2 x 0$. Recall from §3 that $S^2 x B^2$ may be gotten by attaching a 2-handle to the 4-ball along an unknotted circle K with framing zero.



We may obtain τS by giving all the attaching maps of H a full twist (right or left handed as $\pi_1 SO(3) = Z/2Z$) as they pass through a spanning disc for K in S^3. But by Remarks 3.3 (2) and (3), this is also the handlebody structure for CP^2/S' as indicated below



S^4, $S = [K]$ $\qquad$ CP^2, $S' = [K']$ $\qquad$ τS

The reader may verify that the framings on the 2-handles agree, and so $CP^2/S' = \tau S$.

As an immediate consequence, we observe that the homotopy spheres obtained by Gluck's construction "stabilize" (become standard) after blowing up <u>one</u> point. That is

$$\tau S \# \overline{CP}^2 = S^4 \# CP^2$$

Recall that every homotopy 4-sphere stabilizes if we blow up sufficiently many points.

We now invoke Theorem 5.2 together with the previous proposition to deduce that 0-concordant 2-spheres S_0 and S_1 in S^4 yield diffeomorphic homotopy spheres τS_0 and τS_1. In particular

$$\tau(S_0 \# S_1) = \tau S_0 \# \tau S_1$$

and so τ defines a homomorphism from the semigroup of 0-concordance classes of knots in S^4 (under pairwise connected sum) to the semigroup H of homotopy 4-spheres which stabilize after blowing up one point.

Alternatively, this may be deduced directly from Lemma 5.1. In fact we may conclude more. Observe that the only place in the proof of 5.1 where it is essential that we are dealing with a concordance (rather than an arbitrary cobordism) is where we need a particular component of a regular cross section to have self intersection zero. But the regular cross sections of any 0-cobordism between links with zero self intersection automatically have self intersection zero.* Thus Lemma 5.1 holds for such cobordisms, and letting C denote the 0-cobordism classes of knots in S^4 we have

8/99
* total self int = 0 but components need not have self int zero
eg

so needs a different pf. but in S^4 theres no problem

<u>Theorem 6.3</u> τ defines a semigroup homomorphism $\tau : C \to H$

We do not know very much about the semigroup C. If it were trivial, then $\tau S = S^4$ for every knot S in S^4, which would answer a question of H. Gluck (see §1). If it were a group, then we would atleast answer Gluck's question topologically. This follows from the following corollary to 6.3.

<u>Corollary 6.4</u> If S is a knot in S^4 which is invertible in C, then τS is homeomorphic to S^4.

<u>Proof</u> Let S' be an inverse for S in C, that is $S \# S'$ and the unknot S_0 are 0-cobordant. Then

$$
\begin{aligned}
S^4 &= \tau(S_0) \\
&= \tau(S \# S') \\
&= \tau S \# \tau S'
\end{aligned}
$$

The topological Shönflies theorem in S^4 now gives the result.

§7. Handlebody Structure for τS

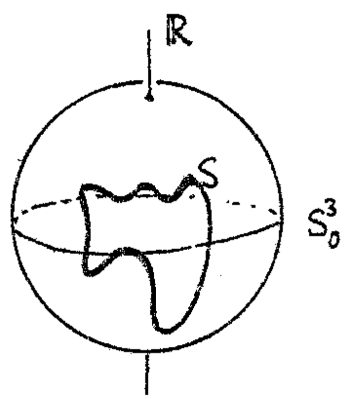
In this section we continue investigating the homotopy 4-sphere τS which arises from Gluck's construction on an embedded 2-sphere S in S^4.

View S^4 as the unit sphere in R^5. Let R^4_t denote the hyperspace $R^4 \times t \subset R^5$. Set $S^3_t = S^4 \cap R^4_t$ and $B^4_t = S^4 \cap (\bigcup_{s \leq t} R^4_s)$, for $0<t<1$. The points $(0,0,0,0,1)$ and $(0,0,0,0,-1)$ will be called the <u>north</u> and <u>south</u> poles of S^4, respectively.

We may adjust S by an isotopy so that

(1) The poles of S^4 do not lie on S

(2) $q|S:S \longrightarrow R$ is a Morse function, where $q:S^4 \longrightarrow R$ is the restriction of the projection $R^5 \longrightarrow R$ onto the last factor.



Such an embedding will be called <u>generic</u>.

It is convenient to introduce a more restrictive class of embeddings. Consider the projection (for any $0 < t < 1$)

$$S^4 - \text{poles} \xrightarrow{p} S^3_t$$

along trajectories of grad(q). We say that a generic $S \subset S^4$ is a <u>critical level embedding</u> if

(1) $p|S$ is a transverse immersion

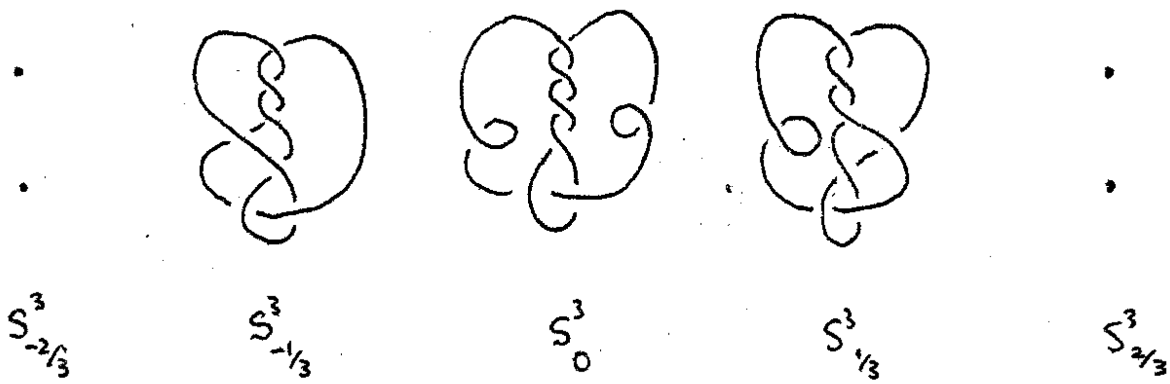
(2) There is a handlebody structure H for S (induced by $q|S$) such that p embeds any union of handles of equal index in H.

In particular, it follows that the projection (under p) of the 1 and 2-handles of H is a ribbon surface of genus zero in S^3_t whose boundary is the unlink.

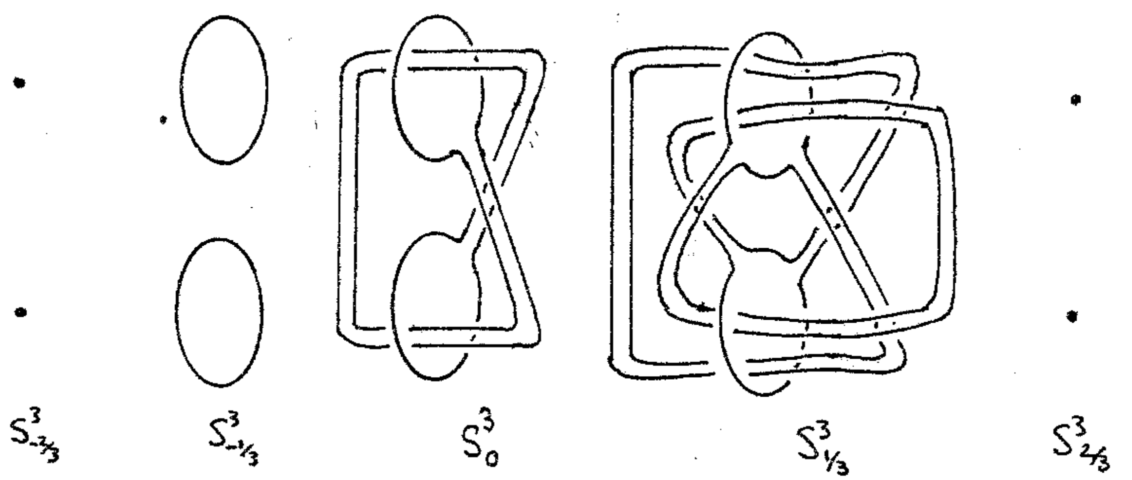
As in §5, we say that S is <u>nice</u> if the critical values of $q|S$ are distinct and increase with the index of their corresponding critical points.

It is well known that any generic embedding $S \subset S^4$ may be moved by an isotopy to a nice critical level embedding. In fact this can be done without changing the number or indices of the critical points of $q|S$.

<u>Example 7.1</u> Consider the knotted 2-sphere shown in crossection below



This is Example 12 in [Fox]. It may be isotoped to the nice critical level embedding shown below



The reader may verify that the two circles in $S^3_{1/3}$ bound disjoint discs.

<u>Proposition 7.2</u> If S has a generic embedding in S^4 with fewer than three local minima, then τS can be built without 3-handles.

<u>Remark</u> The proposition applies to the knot in

Example 7.1, which has two critical points of index zero. We will carry it along to elucidate the proof.

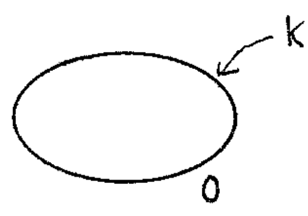
Proof (Sketch) By the remarks above 7.1, we may assume that the inclusion of S in S^4 is a nice critical level embedding.

Fix a handlebody structure for S (as in the definition above) so that $D = B^4_t \cap S$ is a 0-handle, for some t. Setting $B = B^4_t$, it follows that (B, D) is an unknotted ball pair.

Now we may construct in a natural way a handlebody presentation for S^4 with one less 1-handle than the number of 0-handles in S.

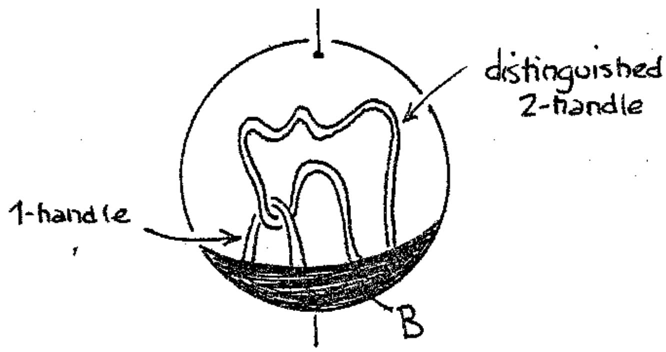
The construction proceeds roughly as follows. We start with B as our 0-handle.

We then add a "distinguished" 2-handle consisting of the part of an appropriate tubular neighborhood of S which lies outside of B. This 2-handle is attached to an unknotted circle K in ∂B.

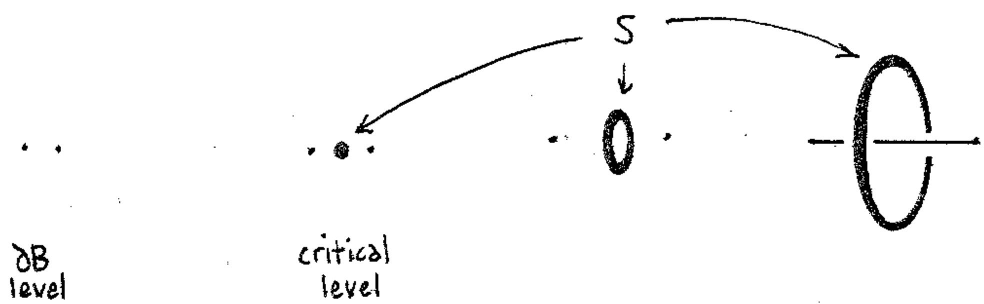


Clearly K represents S (in the sense of §3).

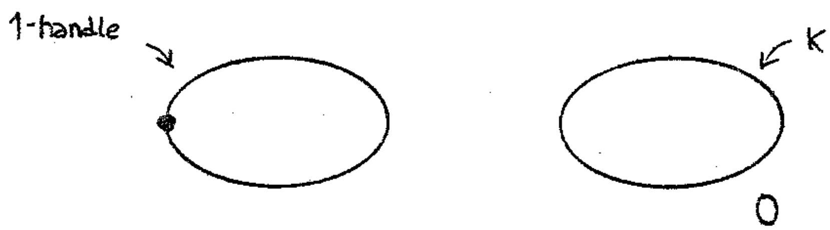
Next we add 1-handles "linking" each 0-handle in S (other than D).



The core of a typical such 1-handle is shown in crossection below (the thin line)

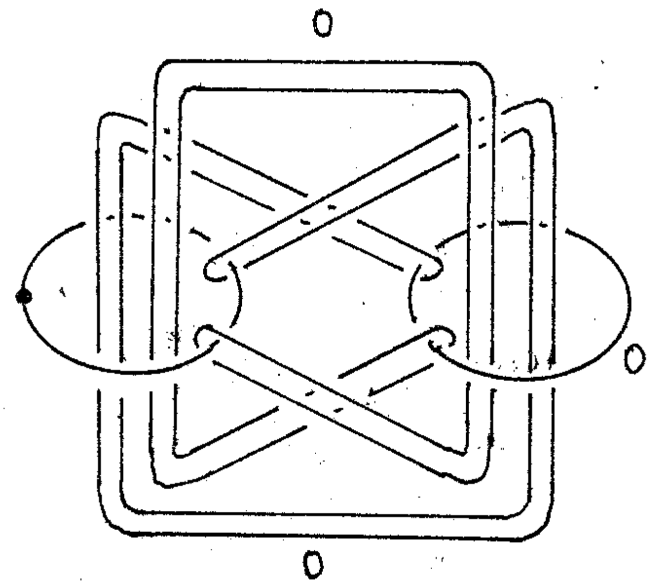


For convenience we denote the attaching map in ∂B by an unknotted circle with a dot on it.



This circle is just a meridian for an arc joining the two points in the attaching sphere (S^0) of the 1-handle. It is understood that any attaching maps which link this circle are actually passing over the 1-handle.

In a similar way we add 2-handles "linking" each 1-handle in S

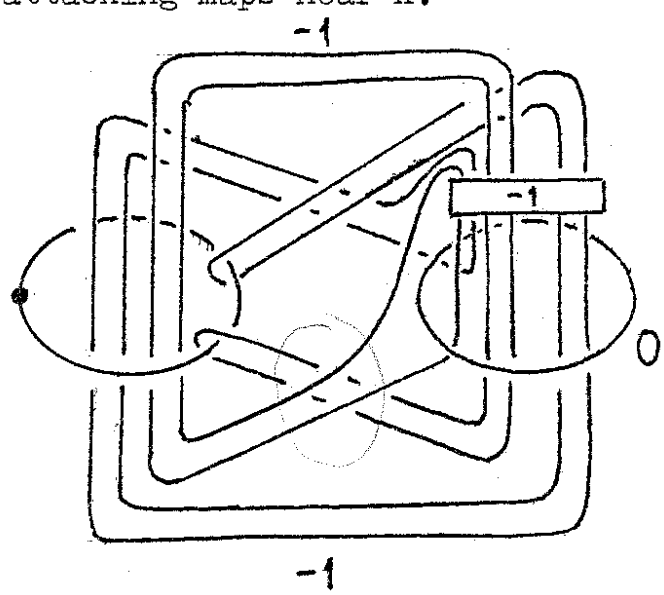


1 and 2-handles for S^4

and 3-handles linking each 2-handle (the picture is harder to draw).

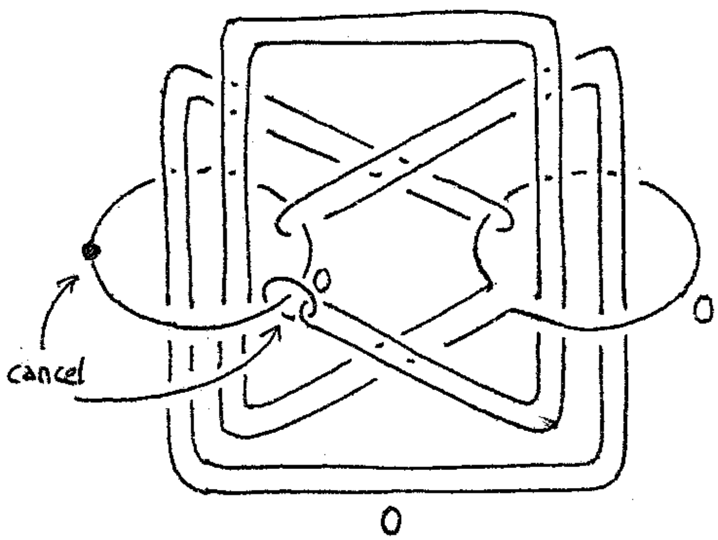
This leaves a 4-ball, which caps off the picture.

The proof of Proposition 6.2 applied to this handlebody presentation of S^4 shows that τS is formed with the same number of handles as S^4, the only difference being the attaching maps near K.



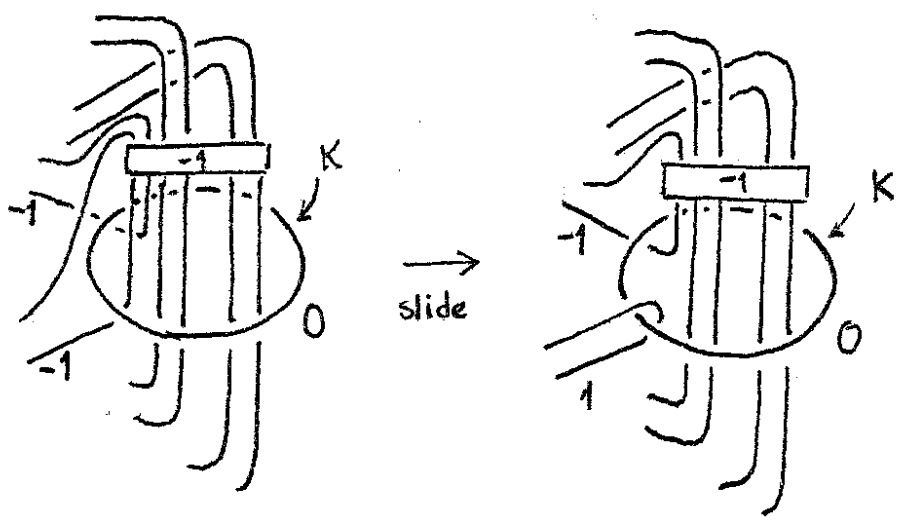
1 and 2-handles for τS

We now note that any 1-handle in S^4 is geometrically cancelled by one of the 2-handles after an appropriate isotopy, as indicated below.



This relies on the fact that the attaching maps of the 2-handles are in the form of ribbons.

Observe that the same thing occurs in τS if we first slide the corresponding 2-handle in τS over K, as shown below



Thus we may cancel one of the 1-handles in τS with a 2-handle, and so τS can be built with two less 1-handles than the number of 0-handles in S. Consequently, if S has fewer than three critical points of index 0, then τS can be built without 1-handles.

Inverting τS, we see that it can be built without 3-handles.

Remark Using the methods of §3, it is possible to give an explicit handlebody presentation for τS with no 3-handles (for S as in 7.2). The 1 and 2-handles then give a presentation for the trivial group $\pi_1 \tau S$.

If this presentation reduces to the trivial presentation by a sequence of extended Nielson transformations, then $(\tau S) \times I = B^5$ [Andrews-Curtis] and so the argument in Corollary 6.4 shows that τS is homeomorphic to S^4.

If this reduction can be realized geometrically (by handle slides in τS) then τS and S^4 are diffeomorphic. In particular, one may show this for the knot in Example 7.1.

References

[Andrews-Curtis] J. Andrews and M. Curtis. Free groups and handlebodies. Proc. A.M.S. 16 (1965), 192-195.

[Barden] D. Barden. h-cobordisms between 4-manifolds. Notes, Cambridge University (1964).

[Cappell-Shaneson] S. Cappell and J. Shaneson. There exist inequivalent knots with the same complement. Annals of Math. 103 (1976), 349-353.

[Cerf] J. Cerf. Sur les difféomorphismes de la sphère de dimension trois ($\Gamma_4 = 0$). Lecture Notes No.53, Springer-Verlag (1968).

[Fox] R. Fox. A Quick Trip Through Knot Theory. Topology of 3-Manifolds and related topics (Fort). Prentice-Hall (1962), 120-167.

[Gluck] H. Gluck. The embedding of two-spheres in the four-sphere. Trans. A.M.S. 104 (1962), 308-333.

[Gordon] C. McA. Gordon. Knots in the 4-sphere are not determined by their complements. To appear in Inventiones.

[Husemöller-Milnor] D. Husemöller and J. Milnor. Symmetric Bilinear Forms. Springer-Verlag, New York (1973).

[Kirby_1] R. Kirby. A calculus for framed links in S^3. To appear in Inventiones.

[Kirby_2] R. Kirby. Problems in low dimensional manifold theory. To appear in Proc. A.M.S. Conference in Topology at Stanford (1976).

[Kirby-Scharlemann] R. Kirby and M. Scharlemann. Eight faces of the Poincaré homology sphere. To appear.

[Kneser] H. Kneser. Geschlossene Flächen in dreidimensionalen Mannigfaltigkeiten. Jahresbericht der Deut. Math. Verein., 38 (1929), 248-260.

[Kreck] M. Kreck. Bordism of diffeomorphisms. Bull. A.M.S. 82 (1976), 759-761

[Rohlin] V. A. Rohlin. New results in the theory of 4-dimensional manifolds. Doklady 84 (1952), 221-224

[Smale] S. Smale. Generalized Poincaré's conjecture in dimensions greater than 4. Annals of Math. 64 (1956), 399-405.

[Spanier] E. H. Spanier. Algebraic Topology. McGraw-Hill (1966).

[Tristram] A. G. Tristram. Some cobordism invariants for links. Proc. Cam. Phil. Soc. 66 (1969), 251-264

[Wall] C. T. C. Wall. On simply-connected four-manifolds. J. London Math. Soc. 39 (1964), 141-149.