ABELIAN INVARIANTS OF SATELLITE KNOTS

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A knot in S^3 whose complement contains an essential** torus is called a satellite knot. In this paper we discuss algebraic invariants of satellite knots, giving short proofs of some known results as well as new results.

To each essential torus in the complement of an oriented satellite knot S, one may associate two oriented knots C and E (the companion and embellishment) and an integer w (the winding number). These are defined precisely below. In the late forties, Seifert [S] showed how to compute the Alexander polynomial of S in terms of w and the polynomials of C and E. Implicit in his work is a description of the Alexander module of S. Shinohara [S1,S2] recovered Seifert's results and computed the signature of S using an illuminating description of the infinite cyclic cover M_S of S (recalled in §1 below) as built up out of the covers of C and E. This description of M_S is in essence also due to Seifert ([S] pp. 25, 28). Using it, Kearton [K] stated (without proof) various properties of the Blanchfield pairing of S, and deduced a formula for the p-signatures of S (obtained independently by Litherland [L] from a 4-dimensional viewpoint).

In §2 we give a complete description of the Blanchfield pairing of S. It depends only on w and the Blanchfield pairings of C and E. (In contrast S cannot be recovered from w, C and E.) In theory one may then compute all the abelian invariants of S from w and the associated invariants of C and E, as the Blanchfield pairing of a knot determines its Seifert form [T2]. This seems difficult in practice however, and so it is appropriate to give more direct computations using the description of M_S. This is done in §3 for the quadratic form of S, which recovers Shinohara's computation of the signature and gives a formula for the rational Witt invariants of S.

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**non-peripheral and incompressible

Notation

Fix an oriented satellite knot S and an essential torus T in the complement of S. Let V denote the solid torus bounded by T. Note that V contains S. The core of V, called the companion of S (associated with T), will be denoted by C. Define the winding number w of S by the homology relation $S \sim wC$ in V. Orient C so that $w \geqq 0$ (there is a choice to be made when $w = 0$). Finally set $E = f(S)$, where $f : V \to S^3$ is an orientation and longitude preserving embedding onto an unknotted solid torus U in S^3. We shall call E the embellishment of S, as S is obtained by embellishing C with E. See Figure 1 for an example. Observe that S cannot be recovered from C, E, and w. One must also know how E lies in U.

1. The Infinite Cyclic Cover

For any oriented knot K in S^3, let N_K be an open tubular neighborhood of K and $X_K = S^3 - N_K$ the exterior of K. Denote the infinite cyclic cover of X_K by M_K, with t_K the canonical covering translation on M_K.

The description of M_S given below is due to Seifert [S] for $w = 0$, and to Shinohara [S2] for $w > 0$. We write nX for the disjoint union of $n \leqq \infty$ copies of a space X.

Theorem 1. There are splittings

$$M_S = M \cup N \quad \text{and} \quad M_E = P \cup Q$$

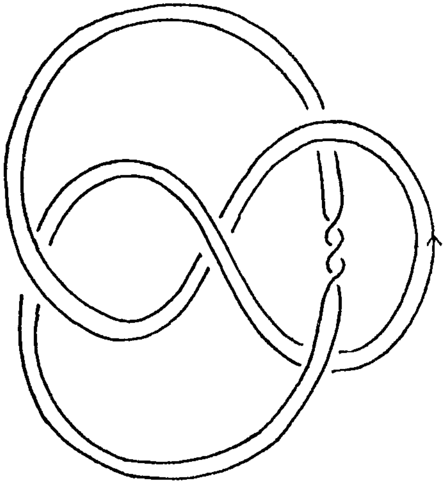
invariant under t_S and t_E, respectively, such that

(1) M, N, P and Q are 3-manifolds with $M \cap N = \partial N$ and $P \cap Q = \partial Q$.

(2) $M = P$, $N = \infty X_C$ (if $w = 0$) or wM_C (if $w > 0$), and $Q = \infty(S^1 \times B^2)$ (if $w = 0$) or $w(\mathbb{R} \times B^2)$ (if $w > 0$).

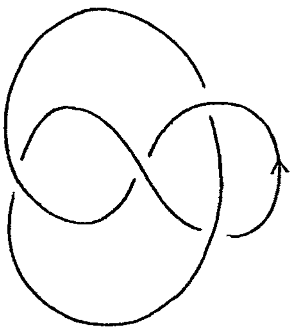
(3) If $w > 0$ then t_S cyclically permutes the components of N. The restriction of t_S^w to any one is t_C.

(4) There is a map $h : M_S \to M_E$ satisfying (a) $ht_S = t_E h$, (b) h maps M homeomorphically onto P, carrying ∂N to ∂Q, and (c) $h(N) = Q$, $h^* : H^2(Q, \partial Q) \to H^2(N, \partial N)$ is an isomorphism with rational coefficients, and $h_* : H_1 N \to H_1 Q$ is an isomorphism for $w = 0$ (with integer coefficients).



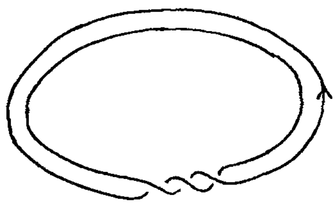
satellite S

(3,2)-cable of the figure 8 knot (w=2)



companion C

figure 8 knot



embellishment E

right-handed trefoil

Figure 1

Proof: For any knot K let $p_K : M_K \to X_K = S^3 - N_K$ denote the covering projection. Choose $N_S \subset V \subset S^3$ and $N_E = f(N_S)$ where $f : V \to U \subset S^3$ is the embedding used to define E. Set $M = p_S^{-1}(S^3\text{-int}V)$, $N = p_S^{-1}(V-N_S)$, $P = p_E^{-1}(S^3\text{-int}U)$, and $Q = p_E^{-1}(U-N_E)$. Then (1) is evident since p_S and p_E are local homeomorphisms.

Observe that there is a Seifert surface F_S for S intersecting $S^3 - \text{int}V$ in w parallel copies of a Seifert surface F_C for C. Let F_E be the Seifert surface for E obtained from $f(F_S \cap V)$ by adjoining w parallel discs in $S^3 - U$. Using F_S, F_E and F_C to construct M_S, M_E and M_C in the usual way, (2) and (3) follow readily. For (4), first extend f to a map $f : S^3 \to S^3$ with $f^{-1}U = V$ and $f^{-1}F_E = F_S$. Now let h be the lift of $f|X_S : X_S \to X_E$. Properties (a), (b) and (c) are easily verified. □

A surgery presentation for M_S can be given which displays the structure in Theorem 1. For example, apply the method of Rolfsen [R] for drawing cyclic covers to C and E separately, while keeping track of X_C. This is illustrated in Figure 2 for the knot of Figure 1.

Remark. A similar description can be given for the finite cyclic covers. Denoting the r-fold cyclic cover of the exterior of a knot K by K^r, and the associated cover of S^3 branched along K by K_r, one has

$$S_r = (E_r - (w,r)(S^1 \times B^2)) \cup (w,r)C^{r/(w,r)}$$

where $(w,r) = \gcd(w,r)$. The proof is analogous to the proof of Theorem 1. It follows by a Mayer-Vietoris argument that

$$H_1S_r = H_1E_r \oplus (w,r)H_1C_{r/(w,r)} .$$

2. The Blanchfield Pairing

Set $\Lambda = Z[t,t^{-1}]$ and $\Lambda_0 = \mathbb{Q}(t)$, the quotient field of Λ. For any oriented knot K, write A_K for the Alexander module of K ($= H_1M_K$ as a Λ-module with $t = (t_K)_*$) and B_K for the Blanchfield pairing on A_K (= linking pairing $A_K \times A_K \to \Lambda_0/\Lambda$). It is well known that A_K is finitely presented with deficiency zero. Thus A_K has a square presentation matrix with entries in Λ. Any such matrix $A_K(t)$ is called an Alexander matrix of K. The associated matrix $B_K(t)$ for B_K (with entries in Λ_0/Λ) is called the associated Blanchfield matrix.

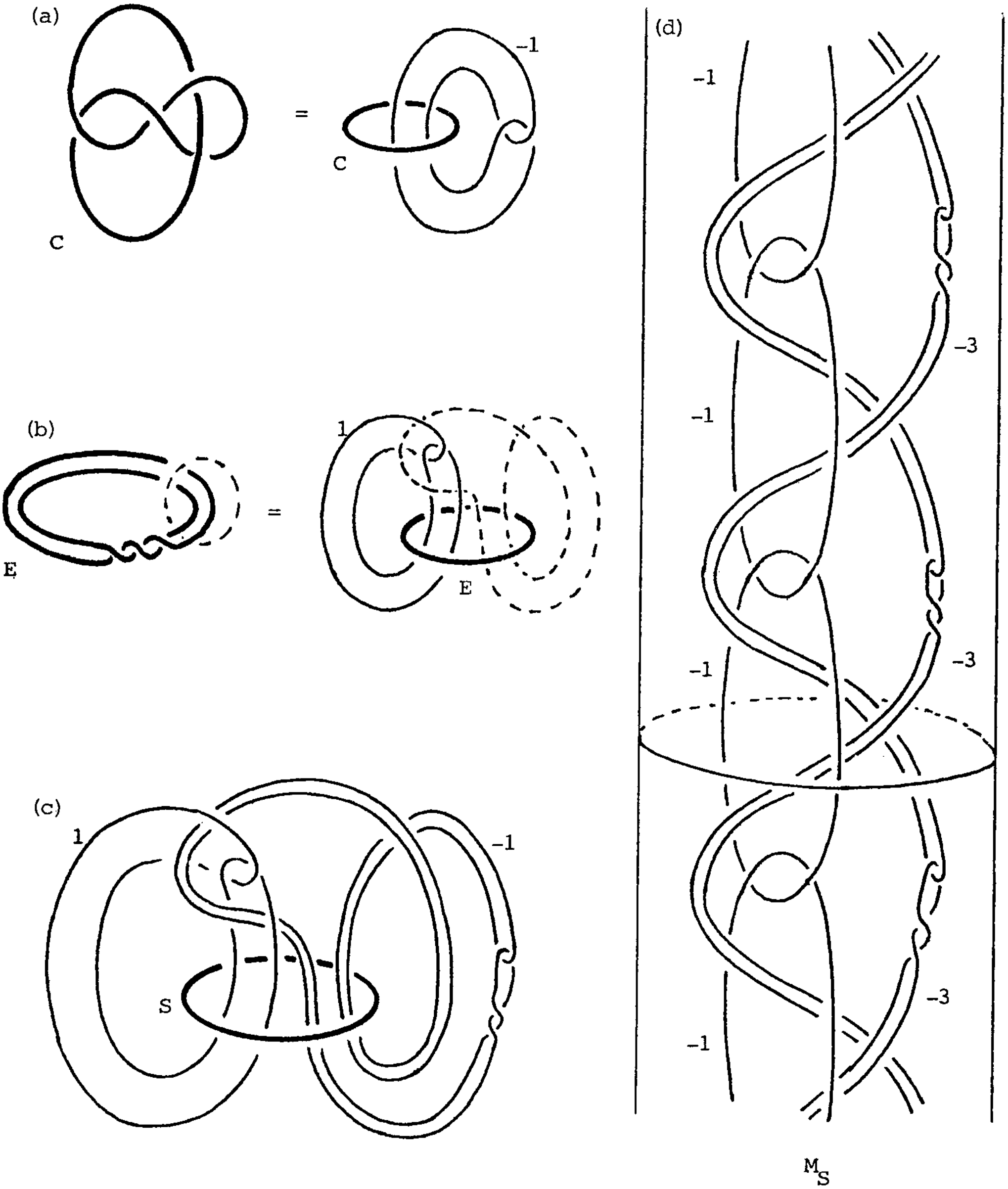


Figure 2

Theorem 2. If $A_E(t)$ and $A_C(t)$ are Alexander matrices for E and C with associated Blanchfield matrices $B_E(t)$ and $B_C(t)$, then

$$A_S(t) = A_E(t) \oplus A_C(t^w)$$

is an Alexander matrix for S with

$$B_S(t) = B_E(t) \oplus B_C(t^w)$$

the associated Blanchfield matrix.

Here $\oplus$ denotes block sum. Since the Alexander polynomial $\Delta_K(t)$ of a knot K is just $\det A_K(t)$, we have

Corollary (Seifert [S]). $\Delta_S(t) = \Delta_E(t)\Delta_C(t^w)$.

Remarks. (1) If $w = 0$, then the theorem simply says

$$A_S(t) = A_E(t) \quad \text{and} \quad B_S(t) = B_E(t) .$$

For, $A_C(t^0) = A_C(1)$ is invertible (as $\Delta_C(1) = 1$) and so the extra generators may be discarded.

(2) The result $A_S(t) = A_E(t) \oplus A_C(t^w)$ was obtained independently by C. Weber [W], and is in fact implicit in the work of Seifert ([S], p.32).

(3) Theorem 2 was used in [LM] to show that algebraic knots (links of isolated singularities of complex curves) are linearly dependent in Levine's algebraic knot concordance group G_-. This contrasts with the fact that torus knots (which are all algebraic) are linearly independent in G_- [L].

Lemma. $H_1M_S = H_1M_E \oplus wH_1M_C$ and $(t_S)_* = (t_E)_* \oplus t$ where $t(x_1,\ldots,x_w) = ((t_C)_*x_w, x_1,\ldots,x_{w-1})$.

Here wG denotes the direct sum of w copies of a group G.

Proof of the Lemma: Adopt the notation of Theorem 1. The map h of (4) induces maps between the Mayer-Vietoris sequences of the triads $(M_S;M,N)$ and $(M_E;P,Q)$, giving

$$\begin{array}{ccccccc}
H_1\partial N & \xrightarrow{i\oplus j} & H_1M \oplus H_1N & \to & H_1M_S & \to & 0 \\
p\downarrow & & q\downarrow \quad r\downarrow & & s\downarrow & & \\
H_1\partial Q & \to & H_1P \oplus H_1Q & \to & H_1M_E & \to & 0 .
\end{array}$$

By (4b), p and q are isomorphisms.

If $w = 0$, then r is an isomorphism by (4c), and so s is also an isomorphism. The action of $(t_S)_*$ follows from (4a).

If $w > 0$, then $j = 0$ and $H_1Q = 0$ by (2). Thus $H_1M_S = H_1M/\mathrm{im}(i) \oplus H_1N = H_1M_E \oplus wH_1M_C$. The action of $(t_S)_*$ follows from (3) and (4a).

Proof of Theorem 2: By the lemma

$$A_S = A_E \oplus A$$

where A is the Λ-module wH_1M_C with t acting as in the lemma. Let $x_1,\ldots,x_m$ and $y_1,\ldots,y_n$ be the generators for A_E and A_C associated with the Alexander matrices $A_E(t)$ and $A_C(t) = (\lambda_{ij}(t))$, respectively.

If $w = 0$, then $A = 0$, and so A_S is presented by

$$A_S(t) = A_E(t)$$

with repsect to the generators $x_1,\ldots,x_m$.

If $w > 0$, consider the generators $Y_i = (y_i, 0,\ldots,0)$ $(i = 1,\ldots,n)$ for A. Evidently

$$\sum_{i=1}^{n} \lambda_{ij}(t^w)Y_i = 0$$

for $j = 1,\ldots,n$. It is easy to verify that any relation in A is a consequence of these, and so A is presented by $A_C(t^w)$ with respect to the Y_i. Thus

$$A_S(t) = A_E(t) \oplus A_C(t^w)$$

presents A_S with respect to the generators $x_1,\ldots,x_m$, $Y_1,\ldots,Y_n$.

It remains to compute $B_S(t)$ with respect to the generators above. Recall that for any knot K, B_K can be computed as follows (see §7 in [G]). Represent x and y, elements of A_K, by cycles c and d in dual t_K-invariant triangulations of M_K. Since A_K is torsion, there is a 2-chain D with $\partial D = \lambda d$ for some λ in Λ. Then

$$B_K(x,y) = \langle c,D\rangle/\lambda$$

where $\langle c,D\rangle = \Sigma(c \cdot t_K^k D)t^k$, k ranging through all integers.

Now if $x = x_i$ and $y = x_j$, it is evident that c and d can be chosen to lie in M (cf. the proof of the lemma above). Since H_1M is Λ-torsion, D can also be chosen in M . Thus the computations of B_S and B_E agree (via the homeomorphism $h : M \to P$ of Theorem 1 (4)). That is

$$B_S(x_i,x_j) = B_E(x_i,x_j) \tag{1}$$

for all i and j between 1 and m . In particular

$$B_S(t) = B_E(t)$$

for $w = 0$.

For $w > 0$, there is more to compute. Since each Y_j can be represented by a cycle in the first copy of M_C ,

$$B_S(x_i,Y_j) = 0 \tag{2}$$

for $i = 1,\ldots,m$ and $j = 1,\ldots,n$. Similarly (or since B_S is Hermitian)

$$B_S(Y_j,x_i) = 0 . \tag{3}$$

Finally, represent Y_i and Y_j by cycles c and d in M_C , with $\lambda(t_S^w)d = \lambda(t_C)d = \partial D$ for some 2-chain D in M_C and $\lambda(t)$ in Λ . Then

$$\begin{aligned} B_S(Y_i,Y_j) &= \Sigma(c \cdot t_S^k D)t^k/\lambda(t^w) \\ &= \Sigma(c \cdot t_C^k D)t^{kw}/\lambda(t^w) \end{aligned} \tag{4}$$

since $c \cdot t_S^k D = c \cdot t_C^{k/w} D$ if k is a multiple of w , and 0 otherwise. But this is just $B_C(y_i,y_j)$ with t replaced by t^w . Thus, combining (1)-(4) gives

$$B_S(t) = B_E(t) \oplus B_C(t^w)$$

for $w > 0$. □

3. The Quadratic Form

Let K be a knot. Trotter [T1] has defined a quadratic form Q_K for K by symmetrizing the Seifert form of K. Q_K is well defined when viewed as an element of the Witt group $W(\mathbb{Q})$ of non-singular rational quadratic forms.

Theorem 3. $Q_S = Q_E$ if w is even, and $Q_S = Q_E + Q_C$ if w is odd (in $W(\mathbb{Q})$).

Remark. Since Q_K is determined by the Blanchfield pairing of K (as are all abelian invariants of K [T2]), Theorem 3 should be in principal a consequence of Theorem 2. It seems difficult, however, to obtain an explicit expression for Q_K from the Blanchfield pairing. Nevertheless, if $w = 0$ then Theorem 2 immediately yields $Q_K = Q_E$, giving Theorem 3 in this case. This case in fact is due to Shinohara [S2].

The proof of the theorem for $w > 0$ uses the following analogue of the lemma in §2. (The statement with $w = 0$ also holds, but is not needed in view of the preceeding remark.)

Lemma. If $w > 0$, then $H^1(M_S, \partial M_S) = H^1(M_E, \partial M_E) \oplus wH^1(M_C, \partial M_C)$ (with rational coefficients) and $t_S^* = t_E^* \oplus t$ where $t(x_1, \ldots, x_w) = (t_C^* x_w, x_1, \ldots, x_{w-1})$.

Proof: Adopt the notation of Theorem 1. The map h of (4) induces maps between the exact sequences of the triples $(M_S, M, \partial M_S)$ and $(M_E, P, \partial M_E)$. Using (2) one has the diagram

$$\begin{array}{ccccccccc} 0 & \to & H^1(M_S,M) & \to & H^1(M_S,\partial M_S) & \to & H^1(M,\partial M_S) & \xrightarrow{\delta} & H^2(M_S,M) \\ & & & & \uparrow & & p\uparrow & & q\uparrow \\ & & 0 & \to & H^1(M_E,\partial M_E) & \to & H^1(P,\partial M_E) & \to & H^2(M_E,P) \end{array}.$$

By (4b) p is an isomorphism. Assuming rational coefficients, q is also an isomorphism, and so $H^1(M_S, \partial M_S) = H^1(M_S, M) \oplus \mathrm{Ker}\,\delta = H^1(N, \partial N) \oplus H^1(M_E, \partial M_E) = wH^1(M_C, \partial M_C) \oplus H^1(M_E, \partial M_E)$. The action of t_S^* follows from (3) and (4a). □

Proof of Theorem 3. Assume $w > 0$, by the remark above. Milnor [M] has shown that for any knot K, Q_K is represented by the form on $H^1(M_K, \partial M_K)$ (with rational coefficients) given by

$$Q_K(x,y) = (t_K^* x)y - x(t_K^* y).$$

It follows from the lemma that

$$Q_S = Q_E + Q$$

where Q is the quadratic form on $H = wH^1(M_C, \partial M_C)$ defined by

$$Q(x,y) = (tx)y - x(ty) .$$

It suffices to show that

$$Q = \begin{cases} 0 & w \text{ even} \\ Q_C & w \text{ odd} \end{cases}$$

in $W(\mathbb{Q})$.

For $i = 1,\ldots,w$, let H_i denote the i^{th} copy of $H^1(M_C, \partial M_C)$ in H , and let x_i in H_i denote the element corresponding to x in $H^1(M_C, \partial M_C)$. Note that $x_i y_j = \delta_{ij} xy$, and $tx_i = x_{i+1}$ (if $i < w$) or $(t_C x)_1$ (if $i = w$) . Set

$$K = \bigoplus_{i \text{ even}} H_i \quad \text{and} \quad L = \bigoplus_{i<w} H_i .$$

If w is even, then Q vanishes on K , and so is a split form. That is, $Q = 0$ in $W(\mathbb{Q})$.

If w is odd, then it is straightforward to verify that $Q|L$ is non-singular (since the cup product on H is) and hyperbolic ($Q|L$ vanishes on K). Thus $Q = Q|L^{\perp}$ in $W(\mathbb{Q})$. But there is an isomorphism $H^1(M_C, \partial M_C) \to L^{\perp}$ associating to x in $H^1(M_C, \partial M_C)$ the element $X = \sum_{i \text{ odd}} x_i + \sum_{i \text{ even}} tx_i$ in $L^{\perp}$. (It is surjective since $\dim(H^1(M_C, \partial M_C)) = \dim(L^{\perp})$.) It is easy to verify that $Q(X,Y) = Q_C(x,y)$ and so $Q_S = Q_E + Q_C$ in $W(\mathbb{Q})$. □

Corollary (Shinohara [S1]). $\sigma S = \sigma E$ if w is even, and $\sigma S = \sigma E + \sigma C$ if w is odd.

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