Evaluations of the 3-Manifold Invariants of Witten and Reshetikhin–Turaev for $sl(2,\mathbf{C})$

ROBION KIRBY AND PAUL MELVIN

In 1988 Witten [W] defined new invariants of oriented 3-manifolds using the Chern–Simons action and path integrals. Shortly thereafter, Reshetikhin and Turaev [RT1] [RT2] defined closely related invariants using representations of certain Hopf algebras $\mathcal{A}$ associated to the Lie algebra $sl(2,\mathbf{C})$ and an r^{th} root of unity, $q = e^{2\pi i m/r}$. We briefly describe here a variant τ_r of the Reshetikhin–Turaev version for $q = e^{2\pi i/r}$, giving a cabling formula, a symmetry principle, and evaluations at $r = 3$, 4 and 6; details will appear elsewhere.

Fix an integer $r > 1$. The 3-manifold invariant τ_r assigns a complex number $\tau_r(M)$ to each oriented, closed, connected 3-manifold M and satisfies:

(1) (multiplicativity) $\tau_r(M\#N) = \tau_r(M)\cdot\tau_r(N)$
(2) (orientation) $\tau_r(-M) = \overline{\tau_r(M)}$
(3) (normalization) $\tau_r(S^3) = 1$

$\tau_r(M)$ is defined as a weighted average of colored, framed link invariants $J_{L,\mathbf{k}}$ (defined in [RT1]) of a framed link L for M, where a coloring of L is an assignment of integers k_i, $0 < k_i < r$, to the components L_i of L. The k_i denote representations of $\mathcal{A}$ of dimension k_i, and $J_{L,\mathbf{k}}$ is a generalization of the Jones polynomial of L at q.

We adopt the notation $e(a) = e^{2\pi i a}$, $s = e(\frac{1}{2r})$, $t = e(\frac{1}{4r})$, (so that $q = s^2 = t^4$), and

$$[k] = \frac{s^{k_i} - \bar{s}^{k_i}}{s - \bar{s}} = \frac{\sin\frac{\pi k}{r}}{\sin\frac{\pi}{r}}\,.$$

DEFINITION: Let

$$\tau_r(M) = \alpha_L \sum_{\mathbf{k}} [\mathbf{k}] J_{L,\mathbf{k}} \tag{4}$$

where α_L is a constant that depends only on r, the number n of components of L, and the signature σ of the linking matrix of L, namely

$$\alpha_L = b^n c^\sigma \stackrel{\text{def}}{=} \left(\sqrt{\frac{2}{r}}\sin\frac{\pi}{r}\right)^n \left(e\left(\frac{-3(r-2)}{8r}\right)\right)^\sigma \tag{5}$$

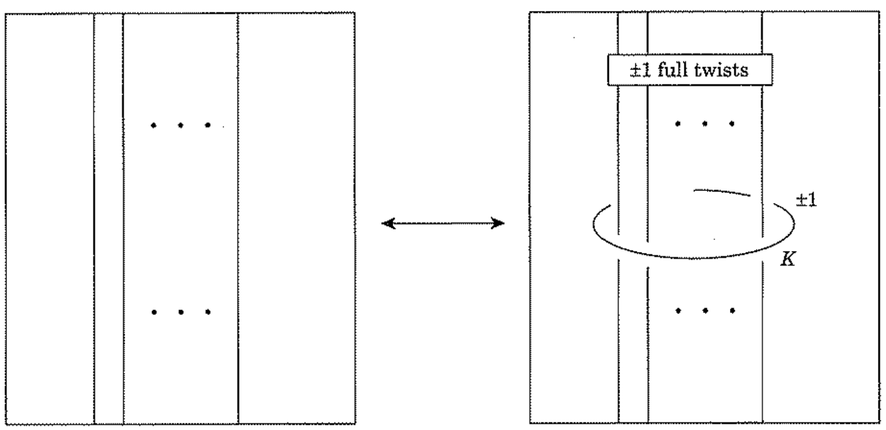
and

$$[\mathbf{k}] = \prod_{i=1}^{n} [k_i]. \tag{6}$$

The sum is over all colorings $\mathbf{k}$ of L.

Remark: The invariant in [**RT2**] also contains the multiplicative factor c^ν where ν is the rank of $H_1(M;Z)$ (equivalently, the nullity of the linking matrix). If this factor is included, then (2) above does not hold, so for this reason and simplicity we prefer the definition in (4).

Recall that every closed, oriented, connected 3-manifold M can be described by surgery on a framed link L in S^3, denoted by M_L [**L1**] [**Wa**]. Adding 2-handles to the 4-ball along L produces an oriented 4-manifold W_L for which $\partial W_L = M_L$, and the intersection form (denoted by $x \cdot y$) on $H_2(W_L;Z)$ is the same as the linking matrix for L so that σ is the index of W_L. Also recall that if $M_L = M_{L'}$, then one can pass from L to L' by a sequence of K-moves [**K1**] [**F-R**] of the form



L_i L'_i

Figure 1

where $L'_i \cdot L'_i = L_i \cdot L_i + (L'_i \cdot K)^2 K \cdot K$.

The constants α_L and $[\mathbf{k}]$ in (4) are chosen so that $\tau_r(M)$ does not depend on the choice of L, i.e. $\tau_r(M)$ does not change under K-moves. In fact, one defines $J_{L,\mathbf{k}}$ (below), postulates an invariant of the form of (4), and then uses the K-move for one strand only to solve uniquely for α_L and $[\mathbf{k}]$. It is then a theorem [**RT2**] that $\tau_r(M)$ is invariant under many stranded K-moves.

To describe $J_{L,\mathbf{k}}$, begin by orienting L and projecting L onto the plane so that for each component L_i, the sum of the self-crossings is equal to the framing $L_i \cdot L_i$.

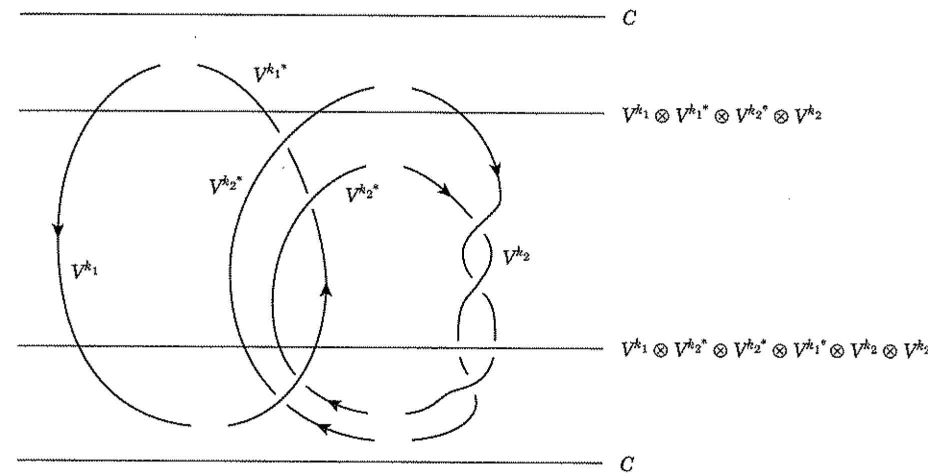


Figure 2

Removing the maxima and minima, assign a vector space V^{k_i} to each downward oriented arc of L_i, and its dual $V^{k_i^*}$ to each upward oriented arc as in Figure 2.

Each horizontal line λ which misses crossings and extrema hits L in a collection of points labeled by the V^{k_i} and their duals, so we associate to λ the tensor products of the vector spaces in order. To each extreme point and to each crossing, we assign an operator from the vector space just below to the vector space just above. The composition is a (scalar) operator from $\mathbf{C}$ to $\mathbf{C}$, and the scalar is $J_{L,\mathbf{k}}$. The vector spaces and operators are provided by representations of $\mathcal{A}$.

To motivate $\mathcal{A}$, recall that the universal enveloping algebra U of $sl(2,\mathbf{C})$ is a 3-dimensional complex vector space with preferred basis X, Y, H and a multiplication with relations $HX - XH = 2X$, $HY - YH = -2Y$ and $XY - YX = H$. To quantize, U, consider the algebra U_h of formal power series in a variable h with coefficients in U, with the same relations as above except that $XY - YX = \frac{\sinh \frac{hH}{2}}{\sinh \frac{h}{2}} = H + \frac{H^3 - H}{24} h^2 + \dots$. Setting $q = e^h$, and then by analogy with the above notation, $s = e^{h/2}$, $t = e^{h/4}$, $\bar{s} = e^{-h/2}$, and $[H] = \frac{s^H - \bar{s}^H}{s - \bar{s}}$, the relations can be written $HX = X(H+2)$, $HY = Y(H-2)$, $XY - YX = [H]$. It is convenient to introduce the element $K = t^H = e^{\frac{hH}{4}}$ and $\bar{K} = \bar{t}^h$. Note that $\bar{K} = K^{-1}$, $KX = sXK$,

$KY = \bar{s}YK$ and $XY - YX = \frac{K^2 - \bar{K}^2}{s - \bar{s}} = [H]$.

We want to specialize U_h at $h = \frac{2\pi i}{r}$ (so $q = e^{2\pi i/r}$) and look for complex representations, but there are difficulties with divergent power series. It seems easiest to truncate, and define $\mathcal{A}$ to be the finite dimensional algebra over $\mathbf{C}$ generated by X, Y, K, $\bar{K}$ with the above relations

$$\begin{aligned} K\bar{K} &= 1 = \bar{K}K \\ KX &= sXK \\ KY &= \bar{s}YK \\ XY - YX &= [H] = \frac{K^2 - \bar{K}^2}{s - \bar{s}} \end{aligned} \tag{7}$$

as well as

$$\begin{aligned} X^r &= Y^r = 0 \\ K^{4r} &= 1. \end{aligned}$$

$\mathcal{A}$ is a complex Hopf algebra with comultiplication Δ, antipode S and counit ε given by

$$\begin{aligned} \Delta X &= X \otimes K + \bar{K} \otimes X \\ \Delta Y &= Y \otimes K + \bar{K} \otimes Y \\ \Delta K &= K \otimes K \qquad (\Delta H = H \otimes 1 + 1 \otimes H) \\ SX &= -sX \\ SY &= -\bar{s}Y \\ SK &= \bar{K} \qquad (SH = -H) \\ \varepsilon(X) &= \varepsilon(Y) = 0 \\ \varepsilon(K) &= 1. \end{aligned} \tag{8}$$

There are representations V^k of $\mathcal{A}$ in each dimension $k > 0$ given by

$$\begin{aligned} Xe_j &= [m + j + 1]e_{j+1} \\ Ye_j &= [m - j + 1]e_{j-1} \\ Ke_j &= s^j e_j \end{aligned} \tag{9}$$

where V^k has basis $e_m, e_{m-1}, \ldots, e_{-m}$ for $m = \frac{k-1}{2}$. The relations in $\mathcal{A}$ are easily verified using the identity $[a][b] - [a+1][b-1] = [a - b + 1]$. For example, the 2 and

3 dimensional representations are

$$X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad K = \begin{pmatrix} t & 0 \\ 0 & \bar{t} \end{pmatrix}$$

$$X = \begin{pmatrix} 0 & [2] & 0 \\ 0 & 0 & [1] \\ 0 & 0 & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & 0 & 0 \\ [1] & 0 & 0 \\ 0 & [2] & 0 \end{pmatrix} \quad \text{and} \quad K = \begin{pmatrix} s & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \bar{s} \end{pmatrix}$$

respectively.

It is useful to represent V^k by a graph in the plane with one vertex at height j for each basis vector e_j, and with oriented edges from e_j to $e_{j\pm 1}$ labeled by $[m \pm j + 1]$ if $[m \pm j + 1] \neq [r] = 0$, indicating the actions of X and Y on V^k. Figure 3 gives some examples, using the identities $[j] = [r - j] = -[r + j]$.

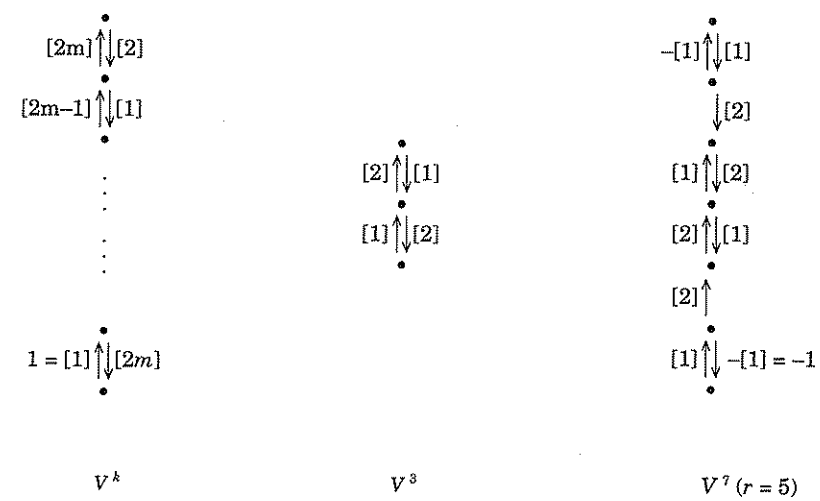


Figure 3

The Hopf algebra structure on $\mathcal{A}$ allows one to define $\mathcal{A}$-module structures on the duals $V^* = \mathrm{Hom}_{\mathbf{C}}(V, \mathbf{C})$ and tensor products $V \otimes W = V \otimes_{\mathbf{C}} W$ of $\mathcal{A}$-modules V and W. In particular, $(Af)(v) = f(S(A)v)$ and $A(v \otimes w) = \Delta A \cdot (v \otimes w)$ for $A \in \mathcal{A}$, $f \in V^*$, $v \in V$, $w \in W$. Thus the vector spaces in Figure 2 will be $\mathcal{A}$-modules and the operators will be $\mathcal{A}$-linear.

The structure of $\mathcal{A}$-modules for $k \leq r$ and their tensor products $V^i \otimes V^j$ for $i+j-1 \leq r$ is parallel to the classical case and is well known:

(10) THEOREM [**RT2**]. *If $k \leq r$, then the representations V^k are irreducible and self dual. If $i+j-1 \leq r$, then $V^i \otimes V^j = \bigoplus_k V^k$ where k ranges by twos over $\{i+j-1,\ i+j-3,\ i+j-5, \ldots, |i-j|+1\}$.*

(11) COROLLARY. *If $k < r$, then*

$$V^k = \sum_j (-1)^j \binom{k-1-j}{j} (V^2)^{\otimes k-1-2j},$$

where the sum is over all $0 \leq j < \frac{k}{2}$.

Here we have written $U = V - W$ to mean $U \oplus W = V$, $jV = \underbrace{V \oplus \cdots \oplus V}_{j \text{ times}}$ and $V^{\otimes j} = \overbrace{V \otimes \cdots \otimes V}^{j \text{ times}}$. This corollary is the key to our later reduction from arbitrary colorings to 2-dimensional ones.

The Hopf algebra $\mathcal{A}$ has the additional structure of a quasi-triangular Hopf algebra [**D**], that is, there exists an invertible element R in $\mathcal{A} \otimes \mathcal{A}$ satisfying

$$\begin{aligned} R\Delta(A)R^{-1} &= \check{\Delta}(A) \quad \text{for all } A \text{ in } \mathcal{A} \\ (\Delta \otimes \mathrm{id})(R) &= R_{13}R_{23} \\ (\mathrm{id} \otimes \Delta)(R) &= R_{13}R_{12} \end{aligned} \tag{12}$$

where $\check{\Delta}(A) = P(\Delta(A))$ and $P(A \otimes B) = B \otimes A$, $R_{12} = R \otimes 1$, $R_{23} = 1 \otimes R$ and $R_{13} = (P \otimes \mathrm{id})(R_{23})$. R is called a *universal R-matrix.* Historically, R-matrices have been found for U_h, $\mathcal{A}$ and other Hopf algebras by Drinfeld [**D**], Jimbo [**J**], Reshetikhin and Turaev [**RT2**] and others. We look for an R of the form $R = \sum c_{nab} X^n K^a \otimes Y^n K^b$, and recursively derive the constants c_{nab} from the defining relation $R\Delta(A)R^{-1} = \check{\Delta}(A)$. This approach was suggested to us by A. Wasserman who had carried out a similar calculation.

(13) THEOREM. *A universal R-matrix for $\mathcal{A}$ is given by*

$$R = \frac{1}{4r} \sum_{n,a,b} \frac{(s-\bar{s})^n}{[n]!} \bar{t}^{\,ab+(b-a)n+n} X^n K^a \otimes Y^n K^b$$

where the sum is over all $0 \leq n < r$ and $0 \leq a,\, b < 4r$ and $[n]! = [n][n-1]\ldots[2][1]$.

(14) COROLLARY. *R acts in the module $V^k \otimes V^{k'}$ by*

$$Re_i \otimes e_j = \sum_n \frac{(s-\bar{s})^n}{[n]!} \cdot \frac{[m+i+n]!}{[m+i]!} \frac{[m'-j+n]!}{[m'-j]!} t^{4ij-2n(i-j)-n(n+1)} e_{i+n} \otimes e_{j-n}$$

where $k = 2m+1$, $k' = 2m'+1$, and $\frac{[p]!}{[n]!} = [p][p-1]\ldots[n+1]$.

EXAMPLES: In $V^2 \otimes V^2$, the R-matrix is

$$(t) \oplus \begin{pmatrix} \bar{t} & \bar{t}(s-\bar{s}) \\ 0 & \bar{t} \end{pmatrix} \oplus (t)$$

with respect to the basis $e_{1/2} \otimes e_{1/2}$, $e_{1/2} \otimes e_{-1/2}$, $e_{-1/2} \otimes e_{1/2}$, and $e_{-1/2} \otimes e_{-1/2}$, and

$$\check{R} = PR = (t) \oplus \begin{pmatrix} 0 & \bar{t} \\ \bar{t} & \bar{t}(s-\bar{s}) \end{pmatrix} \oplus (t). \tag{15}$$

In $V^3 \otimes V^3$,

$$\check{R} = (q) \oplus \begin{pmatrix} 0 & 1 \\ 1 & q-\bar{q} \end{pmatrix} \oplus \begin{pmatrix} 0 & 0 & \bar{q} \\ 0 & 1 & 1-\bar{q} \\ \bar{q} & (q-\bar{q})(1+\bar{q}) & (q-\bar{q})(1-\bar{q}) \end{pmatrix} \oplus \begin{pmatrix} 0 & 1 \\ 1 & q-\bar{q} \end{pmatrix} \oplus (q).$$

It is now possible to assign operators to the following elementary colored tangles [**RT1**]:

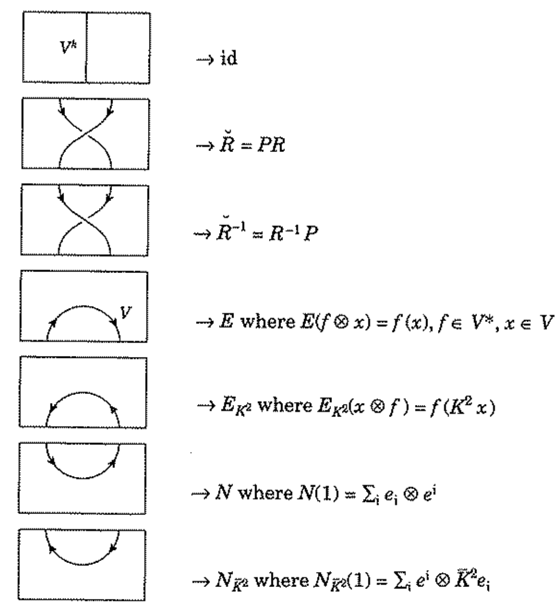


Figure 4

From these elementary tangle operators, define **[RT1]** $\mathcal{A}$-linear operators $J_{T,\mathbf{k}}$ for arbitrary oriented, colored, framed tangles $T, \mathbf{k}$. If T is a link L, then we obtain the scalar $J_{L,\mathbf{k}}$. The invariance of $J_{L,\mathbf{k}}$ under Reidemeister moves on L is well known; the Yang–Baxter equation $(\mathrm{id}\otimes\check{R})(\check{R}\otimes\mathrm{id})(\mathrm{id}\otimes\check{R}) = (\check{R}\otimes\mathrm{id})(\mathrm{id}\otimes\check{R})(\check{R}\otimes\mathrm{id})$ is the key ingredient, and it follows easily from the defining properties (12) for R. Note that $J_{L,\mathbf{k}}$ is independent of choice of orientation of L.

EXAMPLES: The following examples are easily derived from the R-matrix and the irreducibility of V^k:

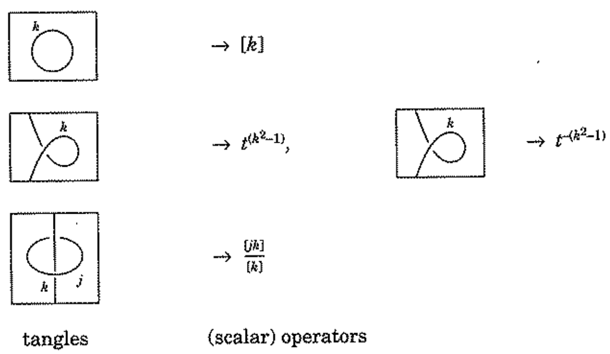


Figure 5

With this definition of $J_{L,\mathbf{k}}$, we have completed the definition of $\tau_r(M_L)$. The examples in Figure 5 can be used to check the one-strand K-move, or conversely, they may be used to solve for the coefficients of $J_{L,\mathbf{k}}$ in the definition of $\tau_r(M_L)$.

When all components of L are colored by the 2-dimensional representation, then $J_{L,2} \stackrel{\mathrm{def}}{=} J_L$ is just a variant of the Jones polynomial. First note that from (15) $\check{R}$ on $V^2 \otimes V^2$ satisfies the characteristic polynomial

$$t\check{R} - \bar{t}\check{R}^{-1} = (s - \bar{s})I.$$

Then, adjusting for framings, J_L satisfies the skein relation

$$qJ_{L_+} - \bar{q}J_{L_-} = (s - \bar{s})J_{L_0} \tag{16}$$

(see **[L2]** for background on skein theory). If $\tilde{V}_L = \tilde{V}(q)$ is the version of the Jones polynomial (for oriented links) satisfying this skein relation and $\tilde{V}_{\mathrm{unknot}} = 1$, then it follows that

$$J_L = J_{L,2} = [2]t^{3L\cdot L}\tilde{V}_L. \tag{17}$$

Remark: the relation between $J_{L,2}$ and $\tilde{V}_L$ is important because the values of $\tilde{V}_L$ at certain roots of unity have a topological description, as they do for the usual Jones polynomial V_L **[LM1]**, **[Lip]**, **[Mur]**. In particular, the values at $q = e(1/r)$, for $r = 1$, 2, 3, 4 and 6, are as follows:

(18)

r	$\tilde{V}_L$	V_L
1	2^{n-1}	$(-2)^{n-1}$
2	$\det L$	$\det L$
3	1	$(-1)^{n-1}$
4	$a\sqrt{2}^{n-1}$	$a(-\sqrt{2})^{n-1}$
6	$\sqrt{3}^{d}(-i)^{\omega}$	$(-\sqrt{3})^{d}(-i)^{\omega}$

where n is the number of components of L, $\det L$ is the value at -1 of the (normalized) Alexander polynomial of L, a is $(-1)^{\mathrm{Arf}(L)}$ when L is proper (so the Arf invariant is defined) and 0 otherwise, d is the nullity of $Q(\mathrm{mod}\,3)$ where Q is the quadratic form of L (represented by $S + S^t$ for any Seifert matrix S of L), and ω is the Witt class of $Q(\mathrm{mod}\,3)$ in $W(\mathbf{Z}/3\mathbf{Z}) = \mathbf{Z}/4\mathbf{Z}$. It is well known that $|\det L| = |H_1(M)|$, where M is the 2-fold branched cover of S^3 along L, and $d = \dim H_1(M;\mathbf{Z}/3\mathbf{Z})$ (since any matrix representing Q is a presentation matrix for $H_1(M)$).

PROPOSITION. *If S is a sublink of L obtained by removing some 1-colored components, then*

$$J_{L,\mathbf{k}} = J_{S,\mathbf{k}|S}. \tag{19}$$

Using (11) and (17) we obtain a formula for the general colored framed link invariant $J_{L,\mathbf{k}}$ in terms of J or $\tilde{V}$ for certain cables of L. In particular, a cabling $\mathbf{c}$ of a framed link L is the assignment of non-negative integers c_i to the L_i, and the associated cable of L, denoted $L^{\mathbf{c}}$, is obtained by replacing each L_i with c_i parallel pushoffs (using the framing).

(20) THEOREM. *Using multi-index notation,*

$$\begin{aligned} J_{L,\mathbf{k}} &= \sum_{\mathbf{j}}(-1)^{\mathbf{j}}\binom{\mathbf{k}-\mathbf{1}-\mathbf{j}}{\mathbf{j}} J_{L^{\mathbf{k}-\mathbf{1}-2\mathbf{j}}} \\ &= [2]\sum_{\mathbf{j}}(-1)^{\mathbf{j}}\binom{\mathbf{k}-\mathbf{1}-\mathbf{j}}{\mathbf{j}} t^{3L^{\mathbf{k}-1-2\mathbf{j}}\cdot L^{\mathbf{k}-1-2\mathbf{j}}}\tilde{V}_{L^{\mathbf{k}-1-2\mathbf{j}}} \end{aligned}$$

for any orientation on $L^{\mathbf{k}-1-2\mathbf{j}}$, where the sum is over all $\mathbf{0} \le \mathbf{j} < \frac{\mathbf{k}}{2}$.

EXAMPLES: If $L = K =$ framed knot, then

$$\begin{aligned} J_{K,3} &= J_{K^2} - 1 \\ J_{K,4} &= J_{K^3} - 2J_K \\ J_{K,5} &= J_{K^4} - 3J_{K^2} + 1. \end{aligned} \tag{21}$$

(22) THEOREM. *$\tau_r(M_L) = \alpha_L \sum_{\mathbf{c}} \langle \mathbf{c} \rangle J_{L^{\mathbf{c}}}$ where the sum is over all cables $\mathbf{c} = (c_1, \ldots, c_n)$, $0 \le c_i \le r-2$, α_L is as in (5), and $\langle \mathbf{c} \rangle = \sum_{\mathbf{j}} [\mathbf{c} + 2\mathbf{j} + 1](-1)^{\mathbf{j}} \binom{\mathbf{c}+\mathbf{j}}{\mathbf{j}}$ where the sum is over all $\mathbf{j} \ge 0$ with $\mathbf{c} + 2\mathbf{j} + \mathbf{1} < r$.*

Remark: a formula like this motivated Lickorish [**L3**] to give an elementary and purely combinatorial derivation of essentially the same 3-manifold invariant as τ_r. The proof reduced to a combinatorial conjecture whose proof has been claimed by Koh and Smolinsky [KS]. This elegant approach is much shorter and simpler. However it may be less useful because the above algebra involving $\mathcal{A}$ organizes a great deal of combinatorial information.

For example, using (20) one can give a recursive formula for $J_{H_n,2} = J_{H_n}$ for the unoriented, n-component, 1-framed, right-handed Hopf link H_n:

$$\begin{aligned} J_{H_0} &= 1 \\ J_{H_1} &= t^3[2] \\ J_{H_n} &= t^{n^2+1}[2n] + \sum_{k=1}^{n/2} (-1)^{k-1} \binom{n-k-1}{k} J_{H_{n-2k}}. \end{aligned} \tag{23}$$

Using deeper properties of $\mathcal{A}$ [**RT2**], one obtains a closed formula:

$$J_{H_n} = t^2 \sum_{k=0}^{n-1} \binom{n-2}{k} [2(n-2k)] t^{(n-2k)^2}. \tag{24}$$

It is not clear how to derive such formulae in a combinatorial way from skein theory.

(25) SYMMETRY PRINCIPLE: Suppose we are given a framed link of $n+1$ components, $L \cup K$, $L = L_1 \cup \cdots \cup L_n$, with colors $\mathbf{l} = l_1 \cup \cdots \cup l_n$ on L and k on K. If we switch the color k to $r - k$, then

$$J_{L \cup K, \mathbf{l} \cup r-k} = \gamma J_{L \cup K, \mathbf{l} \cup k}$$

where $\gamma = i^{ra}(-1)^{\lambda + ka}$, a is the framing on K and $\lambda = \sum_{\text{even } l_i} K \cdot L_i \underset{\text{mod } 2}{\equiv} K \cdot (\mathbf{l} + \mathbf{1})L$.

Use of the Symmetry Principle enables one to cut the number of terms in $\tau_r(M_L)$ from the order of $(r-1)^n$ to $(\frac{r}{2})^n$. It also has interesting topological implications.

EXAMPLE: For $r = 5$ and $L = K$ with framing $a > 0$, then

$$\begin{aligned} \tau_5(M_K) &= \sqrt{\frac{2}{5}} \sin \frac{\pi}{5} e\left(-\frac{9}{40}\right) \sum_{k=1}^{4} [k] J_{K,\mathrm{k}} \\ &= \alpha_K (1 + [2]J_k + [3] i^{5a} (-1)^{2a} J_K + i^{5a}(-1)^a) \\ &= \alpha_K (1 + i^a + ([2] + [3] i^a) J_K) \quad \text{for } a \text{ even} \\ &= \alpha_K (1 + i^a)(1 + [2]^2 t^{3a} \tilde{V}_K) \quad \text{since } [3] = [2] \\ &= 0 \quad \text{for } a \equiv 2 \mod 4. \end{aligned} \tag{26}$$

For $a \not\equiv 2 \mod 4$, this shows that $\tilde{V}_k$ is an invariant of M_K.

Next we discuss the evaluations of $\tau_r(M)$ when $r = 3$, 4 and 6. Note that $\tau_2(M) = 1$.

For $r = 3$,

$$\tau_3(M) = \frac{1}{\sqrt{2}^n} c^\sigma \sum_{S < L} i^{S \cdot S} \tag{27}$$

where $M = M_L$, $c = e\left(-\frac{1}{8}\right) = \frac{1-i}{\sqrt{2}}$ and $<$ denotes sublink and we sum over all sublinks including the empty link ($\phi \cdot \phi = 0$). It is not hard to see how the formula follows from (4) since components with color 1 are dropped (19); it also follows from the cabling formula (22).

Evidently, Formula 27 depends only on the linking matrix A of L. It is not hard to give an independent proof of the well definedness of (27) by checking its invariance under blow ups and handle slides as in the calculus of framed links [**K1**]. This means that $\tau_3(M)$ is an invariant of the stable equivalence class of A (where stabilization means $A \oplus (\pm 1)$). It follows that $\tau_3(M)$ is a homotopy invariant determined by rank $H_1(M; Z)$ and the linking pairing on Tor $H_1(M; Z)$, for these determine the stable equivalence class of A.

The cumbersome sum in (27) can be eliminated by using Brown's $Z/8Z$ invariant β associated with A. View A as giving a $Z/4Z$-valued quadratic form on a $Z/2Z$-vector space by reducing mod 4 along the diagonal (to get the form) and reducing mod 2 (to get the inner product on the vector space). A is stably equivalent to a diagonal matrix and then $\beta = n_1 - n_3 \mod 8$ where n_i is the number of diagonal entries congruent to i (mod 4). Observe that

$$\rho(M) = \sigma - \beta \pmod 8 \tag{28}$$

is an invariant of $M = M_L$.

(29) THEOREM. *If all μ-invariants of spin structures on M are congruent* (mod 4), *then*

$$\tau_3(M) = \sqrt{2}^{b_1(M)} c^{\rho(M)}$$

where $b_1(M) = \operatorname{rank} H_1(M; Z/2Z)$, $c = e\left(-\frac{1}{8}\right)$ *and* $\rho(M)$ *is as above. Otherwise,* $\tau_3(M) = 0$.

(30) COROLLARY. *If M is a $Z/2Z$-homology sphere, then*

$$\tau_3(M) = \pm c^{\mu(M)}$$

where $c = e\left(-\frac{1}{8}\right)$, $\mu(M) = \mu$*-invariant of M and the $\pm$ sign is chosen according to whether* $|H_1(M; Z)| \equiv \pm 1$ or ± 3 (mod 8).

(31) REMARK: $\tau_3(M)$ is not always determined by $H_1(M; Z)$ and the μ-invariants of M (although τ_4 is, see below). For example, if $M = L(4,1) \# L(8,1)$, then $\rho(\pm M) = \pm 2$ so $\tau_3(\pm M) = \pm 2i$, yet M and $-M$ have the same homology and μ-invariants.

(32) THEOREM. $\tau_4(M^3) = \sum_\Theta c^{\mu(M,\Theta)}$ *where* $c = e\left(-\frac{3}{16}\right)$, $\mu(M, \Theta)$ *is the μ-invariant of the spin structure Θ on M and the sum is over all spin structures on M.*

The keys to the proof are these: use the cabling formula (20) to drop 1-colored components, keep 2-colored components and double 3-colored components; the undoubled components turn out to be a characteristic sublink and hence to correspond to a spin structure; at $r = 4$, the Arf invariant (18) comes into play; finally,

$$\mu(M, \Theta) = \sigma - C \cdot C + 8 \operatorname{Arf}(C) \mod 16 \tag{33}$$

is a crucial congruence where C is a characteristic sublink corresponding to Θ.

The congruence (33) is well known in 4-manifold theory [**K2**], being a generalization of Rohlin's Theorem. It turns out, motivated by (32) above that we can give a purely combinatorial proof of (33) without reference to 4-manifolds.

At present, we have no general formula for $\tau_6(M)$ in terms of "classical" invariants of M, although it is plausible that one exists. Indeed, it is immediate from the Symmetry Principle (25) and the cabling formula (20) that $\tau_6(M_L)$ can be expressed in terms of Jones polynomials of cables of L with each component at most doubled. (If the linking number of each component L_i of L with $L - L_i$ is odd, then doubled components may also be eliminated.) Now, since the Jones polynomial of a link at $e\left(\frac{1}{6}\right)$ is determined by the quadratic form of the link (see 18) it would suffice to show that the quadratic forms of these cables are invariants of M.

In particular if M is obtained by surgery on a knot K with framing a, then it can be shown that

$$\tau_5(M) = \frac{(-i)^\sigma}{\sqrt{3}}\left(1 + 2t^{8a} + \frac{3}{2}(1 + (-1)^a)t^{3a}\tilde{V}_K\right)$$

where σ is 0 if $a = 0$, 1 if $a > 0$ and -1 if $a < 0$. It follows that $\tau_6(M)$ is determined by a and the Witt class of the quadratic form Q of K. Thus, for odd a, or $a = 0$, $\tau_6(M)$ is determined by $H_1(M; Z)$ (with its torsion linking form, needed to determine the sign of a when a is divisible by 3). For even a, one also needs to know $H_1(\tilde{M}; Z)$ with its torsion linking form (which determines the Witt class of Q) where $\tilde{M}$ is the canonical 2-fold cover of M.

We are especially grateful to N. Yu. Reshetikhin for his lectures and conversations on [**RT1**] and [**RT2**], and to Vaughn Jones, Greg Kuperberg and Antony Wasserman for valuable insights into quantum groups.

REFERENCES

[**D**] V. G. Drinfel'd, *Quantum groups*, Proc. Int. Cong. Math. 1986 (Amer. Math. Soc. 1987), 798–820.

[**FR**] R. Fenn and C. Rourke, *On Kirby's calculus of links*, Topology **18** (1979), 1–15.

[**J**] M. Jimbo, *A q-difference analogue of $U(\mathcal{G})$ and the Yang-Baxter equation*, Letters in Math. Phys. **10** (1985), 63–69.

[**K1**] R. C. Kirby, *A calculus for framed links in S^3*, Invent. Math. **45** (1978), 35–56.

[**K2**] ———, "The Topology of 4-Manifolds," Lect. Notes in Math., v. 1374, Springer, New York, 1989.

[**KM**] R. C. Kirby and P. M. Melvin, *Evaluations of new 3-manifold invariants*, Not. Amer. Math. Soc., **10** (1989), p. 491, Abstract 89T-57-254.

[**KS**] H. K. Ko and L. Smolinsky, *A combinatorial matrix in 3-manifold theory*, to appear Pacific J. Math..

[**L1**] W. B. R. Lickorish, *A representation of orientable, combinatorial 3-manifolds*, Ann. Math. **76** (1962), 531–540.

[**L2**] ———, *Polynomials for links*, Bull. London Math. Soc. **20** (1988), 558–588.

[**L3**] ———, *Invariants for 3-manifolds from the combinatorics of the Jones polynomial*, to appear Pacific J. Math.

[LM1] W. B. R. Lickorish and K. C. Millett, *Some evaluations of link polynomials*, Comment. Math. Helv. **61** (1986), 349–359.

[Lip] A. S. Lipson, *An evaluation of a link polynomial*, Math. Proc. Camb. Phil. Soc. **100** (1986), 361–364.

[Mur] H. Murakami, *A recursive calculation of the Arf invariant of a link*, J. Math. Soc. Japan **38** (1986), 335–338.

[RT1] N. Yu. Reshetikhin and V. G. Turaev, *Ribbon graphs and their invariants derived from quantum groups*, MSRI preprint (1989).

[RT2] ———, *Invariants of 3-manifolds via link polynomials and quantum groups*, to appear Invent. Math.

[Wa] A. H. Wallace, *Modifications and cobounding manifolds*, Can. J. Math. **12** (1960), 503–528.

[W] Ed Witten, *Quantum field theory and the Jones polynomial*, Comm. Math. Phys. **121** (1989), 351–399.

Department of Mathematics
University of California
Berkeley, CA 94720

Department of Mathematics
Bryn Mawr College
Bryn Mawr, PA 19010