# REAL ANALYSIS II

PAUL MELVIN
BRYN MAWR COLLEGE, SPRING 2016

Real Analysis I was a rigorous treatment of single variable calculus. We began with a study of the real numbers $\mathbb{R}$, sequences of real numbers and their limits (given precisely in terms of the $\varepsilon$-$N$ definition), Cauchy sequences (and the fact that they converge), and various kinds of subsets $X$ of $\mathbb{R}$ (open, closed, bounded, compact and connected). We discussed continuity (via the $\varepsilon$-$\delta$ definitions, and also in terms of sequences and open sets) and uniform continuity of functions $f : X \to \mathbb{R}$, proving the Extreme and Intermediate Value Theorems, and studied the notions of differentiability and Riemann integrability of functions.† After proving the Mean Value Theorem, we established the Fundamental Theorem of Calculus, relating these notions.

Real Analysis II will treat the "rest" of calculus: infinite series, sequences and series of functions (including power series and Fourier series), a little Lebesgue integration theory, a glimpse of functional analysis, some multivariable calculus, and some applications. But first, just for fun, we revisit the "popcorn function" $\mathcal{P} : \mathbb{R} \to \mathbb{R}$ from Real Analysis I, given by

$$\mathcal{P}(x) \;=\; \begin{cases} 1/q & \text{if } x \text{ is rational, } x = p/q \text{ in lowest terms with } q > 0 \\ 0 & \text{if } x \text{ is irrational.} \end{cases}$$

$\mathcal{P}$ is continuous exactly at the irrationals $\mathbb{I}$, i.e. $C_{\mathcal{P}} = \mathbb{I}$, raising the question of whether there exist 'complementary' functions $\mathcal{Q}$, with $C_{\mathcal{Q}} = \mathbb{Q}$. The answer is "no", as first proved by Vito Volterra at the age of 21:

**Volterra's Theorem (1881).** *If $f, g : \mathbb{R} \to \mathbb{R}$ are functions with $C_f$ and $C_g$ dense, then $C_f \cap C_g$ is nonempty (in fact dense).*

*Proof.* We will show that $C_f \cap C_g \cap I \neq \varnothing$ for any open interval $I$. Let $\varepsilon > 0$ and $p \in I \cap C_f$. Then for some closed interval $J \subset I$ about $p$, the image $f(J)$ lies in an interval of length $\varepsilon$, written $\operatorname{diam} f(J) \leq \varepsilon$. Now choosing $q \in \operatorname{int}(J) \cap C_g$, there is a closed interval $K \subset J$ about $q$ such that $\operatorname{diam} g(K) \leq \varepsilon$. Applying this argument repeatedly for $\varepsilon = 1, 1/2, 1/3, \ldots$ yields a nested sequence of closed intervals $I_n \subset I$ such that $\operatorname{diam} f(I_n) < 1/n$ and $\operatorname{diam} g(I_n < 1/n$. It follows that $f$ and $g$ are both continuous on the nonempty intersection $I_1 \cap I_2 \cap I_3 \cap \cdots$. □

## 1. Convergence of sequences of functions Exercises 17 (2, 3, 7–9)

What does it mean for a sequence of functions $f_n : X \to \mathbb{R}$ to converge to a function $f : X \to \mathbb{R}$, written $f_n \to f$? There are many inequivalent ways to define this notion. The two most important are pointwise convergence $f_n \xrightarrow[p]{} f$ meaning $f_n(x) \to f(x)$ for each $x \in X$, i.e.

$$\text{for all } x \in X \text{ and } \varepsilon > 0,\ \exists N \;:\; n \geq N \Longrightarrow |f_n(x) - f(x)| < \varepsilon\,,$$

and uniform convergence $f_n \xrightarrow[u]{} f$ meaning

$$\text{for all } \varepsilon > 0,\ \exists N \;:\; n \geq N \Longrightarrow |f_n(x) - f(x)| < \varepsilon \text{ for all } x \in X.$$

The difference is that for uniform convergence the $N$ may be chosen independent of $x$, while for pointwise convergence it may depend on $x$. Thus uniform convergence implies pointwise convergence, but not conversely, as we shall see.

† Recall that $f$ is integrable on $[a, b]$ if and only if $[a, b] - C_f$ has measure zero, where $C_f$ denotes the set of points where $f$ is continuous.

Geometrically, $f_n \underset{u}{\longrightarrow} f$ means that given any $\varepsilon > 0$, the entire graph of $f_n$ lies within an $\varepsilon$-width band about the graph of $f$ for all sufficiently large $n$. This condition can be restated analytically using the notion of the sup norm

$$\|g\| \;=\; \sup_{x\in X} |g(x)|$$

of a function $g\colon X \to \mathbb{R}$; if $X$ is compact and $g$ is continuous, this is just the maximum value of $|g|$ on $X$, but in general it might be infinite. Since the (vertical) distance between the graphs of $f_n$ and $f$ is $\|f_n - f\|$, it follows that

$$f_n \underset{u}{\longrightarrow} f \iff \|f_n - f\| \longrightarrow 0\,.$$

**Examples** ① The power functions $f_n(x) = x^n$ (where unless otherwise stated, we assume the domain and codomain are both $\mathbb{R}$) do not converge pointwise, since for example $f_n(2) = 2^n$ diverges. But their restrictions $g_n = f_n|[0,1]$ converge pointwise to the function

$$g(x) \;=\; \begin{cases} 0 & \text{if } 0 \le x < 1 \\ 1 & \text{if } x = 1\,, \end{cases}$$

This convergence is not uniform since $\|g_n - g\| \longrightarrow 1$ (draw the picture). Note that the $g_n$'s are all differentiable, whereas $g$ is not even continuous. This shows that *the pointwise limit of a sequence of continuous functions need not be continuous*, and the same is true for differentiable functions. What about (Riemann) integrable functions? Well, the $g_n$'s above and their limit $g$ are all integrable, but consider the following example.

② Enumerate $\mathbb{Q} = \{q_1, q_2, \dots\}$, and set $Q_n = \{q_1, \dots, q_n\}$. The characteristic functions $\chi_{Q_n}$ are all integrable on any closed interval, and converge pointwise to the nonintegrable characteristic function $\chi_{\mathbb{Q}}$. This shows that *the pointwise limit of a sequence of integrable functions need not be integrable.*† But *even if it is integrable, its integral need not be the limit of the integrals of the functions in the sequence*; you are asked to give such an example in HW 17.2 . Hint: Use bump functions with integral 1 that are supported on small intervals.

Continuity and integrability behave better under uniform convergence, but differentiability still misbehaves a bit:

**1.1 Theorem** *Let $f_n\colon X \to \mathbb{R}$ converge uniformly to $f\colon X \to \mathbb{R}$.*

**a)** *If the $f_n$ are continuous, then so is $f$.*

**b)** *If the $f_n$ are integrable on $[a,b]$, then so is $f$, and $\int_a^b f_n \to \int_a^b f$.*

**c)** *If the $f_n$ are differentiable, then $f$ need not be, and even if it is, the sequence $f_n'$ of derivatives need not converge to $f'$. However, if the $f_n$ are differentiable and converge to $f$, and in addition $f_n'$ converges uniformly to some function, then $f$ is differentiable and $f_n' \underset{u}{\longrightarrow} f'$.*

**Remark** In Morgan's text, 1.1a is Theorem 17.3 and 1.1b generalizes Theorem 17.5. In HW 17.9 you are asked to prove the first part of 1.1c by producing an example; we won't prove the last part here.

*Proof.* **a)** (the "$\varepsilon/3$ argument") Given $\varepsilon > 0$, choose $n$ so that $|f_n(x) - f(x)| < \varepsilon/3$ for all $x \in X$, i.e. $\|f_n - f\| < \varepsilon/3$. Now for any $p \in X$, there is a $\delta > 0$ such $|f_n(x) - f_n(p)| < \varepsilon/3$ for

† Replacing $\chi_{Q_n}$ by suitable bump function, with $n$ peaks of height 1 at the points in $Q_n$, shows that *the pointwise limit of a sequence of continuous (even differentiable) functions need not be integrable.*

any $x \in X \cap (p-\delta, p+\delta)$, since $f_n$ is continuous, and so for any such $x$,

$$\begin{aligned} |f(x) - f(p)| &\le |f(x) - f_n(x)| + |f_n(x) - f_n(p)| + |f_n(p) - f(p)| \\ &< \varepsilon/3 + \varepsilon/3 + \varepsilon/3 \;=\; \varepsilon \end{aligned}$$

by the triangle inequality. Thus $f$ is continuous at $p$, and thus at every point in $X$.

**b)** First we show that $f$ is integrable. Let $(P_n, P_n^*)$ be a sequence of partition-sample pairs of $[a,b]$ for which $|P_n| \to 0$. It suffices to show that the sequence $\mathcal{R}_n = \mathcal{R}(f, P_n, P_n^*)$ is Cauchy. But we know that the corresponding sequence $\mathcal{R}_{mn} = \mathcal{R}(f_m, P_n, P_n^*)$ converges for each $m$, since $f_m$ is integrable, and the uniform convergence $f_m \underset{u}{\longrightarrow} f$ shows that $|\mathcal{R}_{mn} - \mathcal{R}_n| < \varepsilon$ for $m$ large enough so that $\|f_m - f\| < \varepsilon/2(b-a)$, and the result follows from the triangle inequality.

Now that we know that $f$ is integrable, we compute

$$\left| \int_a^b f_n - \int_a^b f \right| = \left| \int_a^b (f_n - f) \right| \le \int_a^b \|f_n - f\| \longrightarrow 0$$

to show that $\int_a^b f_n \to \int_a^b f$. □

<u>**Example**</u> $f_n(x) = x^2 + e^{x^2}/n$ converge pointwise but not uniformly to $f(x) = x^2$, since $\lim_{n\to\infty} e^{x^2}/n = 0$ for any given $x$ while $\lim_{x\to\infty} e^{x^2}/n = \infty$ for any given $n$. But when restricted to $[0,1]$, the convergence becomes uniform (since on $[0,1]$, $\|f_n - f\| \le e/n \to 0$). Thus

$$\lim_{n\to\infty} \int_0^1 (x^2 + e^{x^2}/n)\, dx \;=\; \int_0^1 \lim_{n\to\infty} (x^2 + e^{x^2}/n)\; dx \;=\; \int_0^1 x^2\, dx \;=\; 1/3.$$

<u>**Remark**</u> Although pointwise convergence $f_n \underset{p}{\longrightarrow} f$ does not imply uniform convergence in general, it does provided the functions $f_n$ are *nice enough*, for example <u>equicontinuous</u>, meaning for every $\varepsilon > 0$, there is a $\delta > 0$ that can be chosen *independent of* $n$ such that $|x - y| < \delta \Longrightarrow |f(x) - f(y)| < \varepsilon$. In particular, this implies that the $f_n$ are uniformly continuous, but it is stronger than that. An instance of this is explored in HW 17.8.

## <u>2. The Lebesgue theory</u> Exercises 18 (2, 5, 9)

As noted above, the Riemann integral does not behave well for pointwise convergent sequences of functions, but the more general <u>Lebesgue integral</u> does. It is defined for a wider class of functions than the Riemann integral, the (Lebesgue) <u>measurable functions</u>, and gives the same answer when integrating a Riemann integrable function. The measurable functions include, for example, the characteristic function $\chi_{\mathbb{Q}}$ of the rationals, and more generally $\chi_S$ for any (Lebesgue) <u>measurable set</u> $S$. Changing the values of any measurable function on a measure zero subset of its domain produces another measurable function with the same integral, so in particular, measurable functions need not be bounded. We also allow infinite valued integrals and integrals over unbounded domains, and thus include the improper integrals studied in calculus.

<u>**Measurable sets**</u> Given an arbitrary subset $S$ of $\mathbb{R}$, consider a cover of $S$ by countably many intervals. This cover has a <u>total length</u> (the possibly infinite sum of the lengths of all its intervals) and the infimum of the total lengths of all such covers is called the <u>outer measure</u> of $S$, denoted $\mu^*(S)$. We say that $S$ is <u>measurable</u> if $\mu^*(A) = \mu^*(A \cap S) + \mu^*(A - S)$ for every $A \subset \mathbb{R}$, and in that case we define the <u>measure</u> $\mu(S)$ of $S$ to be its outer measure. It can be shown that the measurable sets include all open and closed sets, but there are many others. If one assumes the "axiom of choice", however, then there *do* exist non-measurable sets. There are analogous definitions for subsets of $\mathbb{R}^n$. See the Wikipedia articles on Lebesgue measure and the Vitali set for more details.

**Measurable functions and the Lebesgue integral** A function $f : X \to \mathbb{R}$ is measurable if $X$ is measurable and $f^{-1}(U)$ is measurable for every open subset $U$ of $\mathbb{R}$, cf. the Wikipedia article on Measurable functions.

Now given *any* function $f : X \to \mathbb{R}$, where $X \subset \mathbb{R}$, let $\mathcal{R}_+$ be the region above $X$ and below the graph of $f$, and $\mathcal{R}_-$ be the region below $X$ and above the graph of $f$. Then it can be shown that $f$ is measurable if and only if both $\mathcal{R}_+$ and $\mathcal{R}_-$ are measurable sets. In that case the (Lebesgue) integral of $f$ over $X$ is defined by

$$\int_X f \;:=\; \mu(\mathcal{R}_+) - \mu(\mathcal{R}_-)$$

Another way of explaining this (see Wikipedia's article on Lebesgue integration) is that the Riemann integral computes (or attempts to compute) this signed measure by partitioning the domain, thus adding it up "by vertical columns". In contrast, the Lebesgue integral computes it by partitioning the codomain, thus adding it up "by horizontal rows". In particular, if $f$ is non-negative (i.e. $f(x) \geq 0$ for all $x \in X$) then

$$\int f\, d\mu \;=\; \int_0^\infty f^*(t)\, dt$$

where $f^*(t) = \mu(\mathcal{R} \cap \{y = t\})$, and the right hand side is an improper Riemann integral.

**Properties of the Lebesgue integral** There are many useful theorems that specify when one can switch the order of taking limits, integrating and differentiating. Below are two of the most important ones, stated without proof but with examples to illustrate their application. The first concerns interchanging limits and integration.

**2.1 Lebesgue's Dominated Convergence Theorem** (DCT) *Let $f_n : X \to \mathbb{R}$ be a pointwise convergent sequence of measurable functions that are dominated by a measurable function $g$ (meaning $|f_n(x)| < g(x)$ for all $n$ and all $x \in X$) for which $\int_X g$ is finite. Then*

$$\lim_{n\to\infty} \int_X f_n \;=\; \int_X \lim_{n\to\infty} f_n.$$

**Examples** 1 (Morgan 18.1) Compute $\displaystyle\lim_{n\to\infty} \int_1^2 x^{2-\sin nx}/n\, dx$.

Solution The integrands are dominated on $[1, 2]$ by $x^3$, since $|\sin nx| \leq 1$, so by the DCT

$$\lim_{n\to\infty} \int_1^2 x^{2-\sin nx/n}\, dx \;=\; \int_1^2 (\lim_{n\to\infty} x^{2-\sin nx/n})\, dx \;=\; \int_1^2 x^2 \;=\; 5/3.$$

2 (Morgan 18.3) Compute $\displaystyle\lim_{n\to\infty} \int_0^1 \frac{1}{nx}\, dx$.

Solution The DCT does not apply here: If the integrands were dominated on $[0, 1]$ by some function *with finite integral*, then we would compute

$$\lim_{n\to\infty} \int_0^1 \frac{1}{nx}\, dx \;=\; \int_0^1 (\lim_{n\to\infty} \frac{1}{nx})\, dx \;=\; \int_0^1 0 \;=\; 0,$$

but this is wrong. Indeed from Calculus I, we compute

$$\int_0^1 \frac{1}{nx}\, dx \;=\; \lim_{a\to 0} \frac{1}{n} \log x \Big|_a^1 \;=\; \infty$$

and so in fact $\displaystyle\lim_{n\to\infty} \int_0^1 \frac{1}{nx}\, dx = \infty$.

The second theorem we state concerns the process of writing integrals of measurable functions $f : X \to \mathbb{R}$ of two variables (so $X \subset \mathbb{R}^2$) as iterated integrals.

**2.2 <u>Fubini's Theorem</u>** (<u>FT</u>) *If $f : X \to \mathbb{R}$ is non-negative valued, then $\iint_X f$ can be computed as an iterated integral <u>in</u> <u>either</u> <u>order</u>.* (This first statement is also known as Tonelli's Theorem.) *The same is true for arbitrary $f$ provided $\iint_X |f|$ is finite.*

Thus if one can show $\iint_X f$ is finite by estimating $|f|$ and/or integrating using <u>either</u> order of integration, then one can evaluate it using whichever order of integration is more convenient.

**<u>Example</u>** ① (Morgan 18.4) Compute $\displaystyle\int_0^{10}\int_0^{\pi/3} xy\cos xy^2\,dx\,dy$: Noting that

$$\int_0^{10}\int_0^{\pi/3} |xy\cos xy^2|\,dx\,dy \;<\; \int_0^{10}\int_0^{\pi/3} xy\,dx\,dy \;=\; (10\pi/3)^2 \;<\; \infty$$

we compute

$$\begin{aligned}\int_0^{10}\int_0^{\pi/3} xy\cos xy^2\,dx\,dy \;&=\; \int_0^{\pi/3}\int_0^{10} xy\cos xy^2\,dy\,dx \;=\; \int_0^{\pi/3}\frac{1}{2}\sin 100x \\ &=\; -\frac{1}{200}\cos 100x\Big|_0^{\pi/3} \;=\; \frac{1}{400}+\frac{1}{200} \;=\; \frac{3}{400}.\end{aligned}$$

② (Morgan 18.5) Compute $\displaystyle\int_0^1\int_y^1 \frac{x^2}{y^2}\,e^{-x^2/y}\,dx\,dy$, for homework.[†]

③ We will use Fubini's Theorem to show that the double integral $I = \iint_S |f|$, where $S$ is the unit square $[0,1]\times[0,1]$ and

$$f(x,y) \;=\; \begin{cases}\dfrac{x^2-y^2}{(x^2+y^2)^2} & \text{for } (x,y)\neq(0,0)\\ 0 & \text{for } (x,y)=(0,0)\end{cases}$$

is infinite. Set $I_{xy} = \int_0^1\int_0^1 f\,dx\,dy$ and $I_{yx} = \int_0^1\int_0^1 f\,dx\,dy$. Since $f(x,y) = -f(y,x)$, we have $I_{xy} = -I_{yx}$. If $I$ were finite, then Fubini's Theorem would imply $I_{xy} = I_{yx}$, and so both $I_{xy}$ and $I_{yx}$ would be zero. But a direct calculation shows $I_{yx} = \pi/4$ :

$$\begin{aligned}\int_0^1 f(x,y)\,dy \;&=\; \int_0^1 \frac{(x^2+y^2)-2y^2}{(x^2+y^2)^2}\,dy \;=\; \int_0^1\frac{1}{x^2+y^2}\,dy + \int_0^1\frac{-2y^2}{(x^2+y^2)^2}\,dy \\ &=\; \int_0^1\frac{1}{x^2+y^2}\,dy + \int_0^1 y\,\frac{d}{dy}\left(\frac{1}{x^2+y^2}\right)dy \qquad \text{(now integrate by parts)} \\ &=\; \int_0^1\frac{1}{x^2+y^2}\,dy + \left(\frac{y}{x^2+y^2}\Big|_{y=0}^{y=1} - \int_0^1\frac{1}{x^2+y^2}\,dy\right) \;=\; \frac{1}{x^2+1}\end{aligned}$$

and so $I_{yx} = \displaystyle\int_0^1 1/(x^2+1)\,dx = \tan^{-1}x\Big|_0^1 = \pi/4$.

Morgan states one other such theorem (Leibnitz's Theorem) that specifies when one can interchange derivatives with integrals, but we will not discuss that here.

[†] Be careful with the bounds of integration when switching the order of integration! Also note that this is an improper integral, since the integrand is undefined when $y = 0$. If one wishes to avoid this, the integrand can be redefined arbitrarily along the $x$-axis without changing the integral (since the $x$-axis has measure zero in $\mathbb{R}^2$).

**3. Infinite Series** Exercises 19 (2, 4, 6, 8, 12)

An infinite series is a sum of the numbers in an infinite sequence $a_1, a_2, a_3, \ldots$

$$\sum_{n=1}^{\infty} a_n \;=\; a_1 + a_2 + a_3 + \cdots$$

The numbers $a_n$ are referred to as the terms of the series. From this series we form the sequence

$$A_k \;:=\; \sum_{n\le k} a_n \;=\; a_1 + \cdots + a_k$$

of partial sums. Thus $A_1 = a_1$, $A_2 = a_1 + a_2$, $A_3 = a_1 + a_2 + a_3$, etc. The series is said to converge or diverge according to whether $A_k$ converges or diverges. If $A_k$ *does* converge, say to $A$, then we say that the series converges to $A$, and simply write $\sum_{n=1}^{\infty} a_n = A$. Sometimes we also write $\sum_{n=1}^{\infty} a_n = \infty$ to mean that the series "diverges to $\infty$", i.e. that $A_k \to \infty$.

When considering convergence or divergence of a series, we may shorten the notation $\sum_{n=1}^{\infty} a_n$ to $\sum a_n$. This is because for any $p$ and $q$,

$$\sum_{n\ge p} a_n \text{ converges } \iff \sum_{n\ge q} a_n \text{ converges}$$

although these "tails" of the original series will likely converge to different numbers. Two important special types of series are the *positive* ones, for which the terms $a_n$ are eventually positive ("eventually" meaning "for all sufficiently large $n$"), and the *alternating* ones, for which the terms eventually alternate in sign.

There is one very basic test for divergence of a general series:

**3.1 Divergence Test** *If the terms in a series* ***do not converge to zero****, then the series diverges.*

You are asked to prove (the contrapositive) of this in the homework: If a series converges, then its terms converge to zero.† Here's an example: $\sum n$ diverges (to $\infty$) since $n \not\to 0$ . But be careful, the converse need not be true, e.g. $1/n \to 0$ but $\sum 1/n$ diverges, as explained below.

Another important property of infinite series is "linearity", which follows immediately from the analogous properties of sequences:

**3.2 Proposition** *If $\sum a_n = a$ and $\sum b_n = b$, then $\sum (a_n + b_n) = a + b$ and $\sum ca_n = ca$.*

**Examples** ① (geometric series) Any series $\sum a_n$ for which there is a number $r$ (the ratio) such that $ra_n = a_{n+1}$ is called a geometric series. Thus a general geometric series is of the form

$$a + ar + ar^2 + \cdots \;=\; \sum_{n=0}^{\infty} ar^n$$

If $|r| \ge 1$ (and $a \ne 0$) then this series diverges (by 3.1), but if $|r| < 1$ then it converges to $s = a/(1-r)$ ("first term over 1 minus the ratio"). The informal proof is to note that $s - rs = a$, and then solve for $s$. The precise proof, starting with $s_n - rs_n = a - a^{r+1}$, then solving for $s_n$ and taking the limit, is spelled out in Morgan's text. An example is

$$\frac{1}{10} + \frac{1}{100} + \frac{1}{1000} + \cdots = \frac{1/10}{1 - 1/10} = 1/9$$

which shows $.111\cdots = 1/9$.

② ($p$-series) For any $p > 0$, the $p$-series is

$$\sum_{n=1}^{\infty} \frac{1}{n^p} \;=\; 1 + \frac{1}{2^p} + \frac{1}{3^p} + \cdots$$

† You may use the following consequence of the fact that Cauchy sequences converge: ***A series converges if and only if its sequence of partial sums is Cauchy***. This is the basis for many proofs in this subject.

which is known diverge if $p \le 1$ but converge if $p > 1$ to a value known as $\zeta(p)$. The function $\zeta : (1, \infty) \to \mathbb{R}$ can be extended to a complex function $\zeta : \mathbb{C} - \{1\} \to \mathbb{C}$ called the Riemann zeta function, which leads to the famous "Riemann Hypothesis" whose solution is worth a million dollars. The values $\zeta(p)$ for $p$ even are known precisely (.g. $\zeta(2) = \pi^2/6$ and $\zeta(4)\pi^4/90$) but other values of $\zeta(p)$ are not so simple.

The case $p = 1$ is everybody's favorite divergent series, the harmonic series:

$$1 + \tfrac{1}{2} + \tfrac{1}{3} + \tfrac{1}{4} + \cdots$$

Why does it diverge? Here's one very simple way to see this: add the terms up in groups as indicated below:

$$1 + \tfrac{1}{2} + (\tfrac{1}{3} + \tfrac{1}{4}) + (\tfrac{1}{5} + \tfrac{1}{6} + \tfrac{1}{7} + \tfrac{1}{8}) + (\tfrac{1}{9} + \tfrac{1}{10} + \tfrac{1}{11} + \tfrac{1}{12} + \tfrac{1}{13} + \tfrac{1}{14} + \tfrac{1}{15} + \tfrac{1}{16}) + \cdots$$

taking the first two by themselves, then the next two as a group, then the next four, eight, etc., doubling the size of the group at each stage. Each group clearly adds up to more than $1/2$, so the total sum is more than $1 + 1/2 + 1/2 + 1/2 + 1/2 + \cdots = \infty$. There's a fancier way to do this, and all the $p$-series at once, using the "integral test" for convergence. This is the first of several important test we learn about in calculus.

## Convergence Tests for Positive Series

Throughout this subsection, assume that the series $\sum a_n$ is positive. Recall that this means $a_n > 0$ for all sufficiently large $n$.

**3.2 Integral Test** *If there is a continuous, decreasing, positive, real-valued function $f$ defined on some positive ray $(a, \infty)$ such that $f(n) = a_n$ for all sufficiently large $n$, then $\sum a_n$ converges if and only if the improper integral $\int_a^\infty f$ converges (i.e. is finite).*

Give the picture proof.

For example if $p > 0$, then applying this test to the function $f_p(x) = x^{-p}$ for $x \in (1, \infty)$ gives the $p$-series result stated above. Indeed, if $p \ne 1$ then

$$\int_1^\infty x^{-p}\,dx \;=\; \frac{1}{1-p}\,x^{1-p}\,\Big|_1^\infty \;=\; \begin{cases} 1/(p-1) & \text{if } p > 1 \\ \text{diverges} & \text{if } p < 1 \end{cases}$$

and if $p = 1$ we have

$$\int_1^\infty \frac{1}{x}\,dx \;=\; \log x\,\Big|_1^\infty \;=\; \infty\,.$$

For the next two tests we compare $\sum a_n$ with any other positive series $\sum b_n$ whose convergence or divergence we know.

**3.2 Comparison Test a)** *If $\sum b_n$ converges and $b_n \ge a_n$ for all $n$, then $\sum a_n$ converges.* **b)** *If $\sum b_n$ diverges and $b_n \le a_n$ for all $n$, then $\sum a_n$ diverges.*†

*Proof.* Note that b) follows from a) by interchanging the two series. For a), let $A_k = \sum_{n \le k} a_n$ and $B_k = \sum_{n \le k} b_n$ be the sequences of partial sums for $\sum a_n$ and $\sum b_n$. Since $\sum b_n$ converges, the sequence $B_k$ is Cauchy. Since Cauchy sequences converge, it suffices to show $A_k$ is Cauchy, and this follows since $|A_\ell - A_m| \le |B_\ell - B_m|$ for any $\ell$ and $m$. □

**3.3 Limit Comparison Test** *If $a_n/b_n$ approaches a positive finite limit $c$ (i.e. $0 < c < \infty$) then $\sum a_n$ converges if and only if $\sum b_n$ converges.*

The proof is essentially the same as the proof of 3.2, and is left for the reader.

† In both parts, one may replace "all $n$" by "all sufficiently large $n$".

**Examples** ① Since $n^2+1>n^2$, we have $1/(n^2+1)<1/n^2$, and so $\sum 1/(n^2+1)$ converges by comparison with $\sum 1/n^2$ (which is a convergent $p$-series, for $p=2$).

② What about $\sum 1/(n^2-1)$? One can still compare this with $\sum 1/n^2$, but not directly. Instead, use the limit comparison test: $(1/(n^2))/(1/(n^2-1)) = (n^2-1)/n^2 \to 1$, so $\sum 1/(n^2-1)$ converges since $\sum 1/n^2$ does.

③ By a similar argument as in ②, if $p(x)$ and $q(x)$ are polynomials of degrees $a$ and $b$, respectively, then $\sum p(n)/q(n)$ converges if $b \geq a+2$, and diverges if $b < a+2$, by comparison with $p$-series for $p=b-a$.

**Convergence Tests for Alternating Series**

**3.4 Alternating Series Test** *If the absolute values of the terms in an alternating series $\sum a_n$ form a **decreasing** sequence converging to zero, then $\sum a_n$ converges.*

*Proof.* Let $A_1, A_2, A_3, \ldots$ denote the sequence of partial sums of $\sum a_n$. Since the $a_n$'s alternate in sign and decrease in absolute value, the even sequence $A_2, A_4, A_6, \ldots$ is monotonic – say increasing – while $A_1, A_3, A_5, \ldots$ is decreasing. Since $a_n \to 0$, it follows that $A_1, A_2, A_3, \ldots$ is Cauchy, and so $\sum a_n$ converges. □

**Example** The alternating harmonic series $\sum_{n=1}^{\infty}(-1)^{n-1}/n$. In fact, it converges to $\log 2$ (give a picture proof).

## 4. Absolute Convergence Exercises 20 (1, 3, 5, 7, 9, 10)

**Definition** An infinite series $\sum a_n$ is said to converge absolutely if the corresponding series of absolute values $\sum |a_n|$ converges.

It is shown below that all absolutely convergent sequences converge, but the converse may fail. For example the alternating harmonic series $\sum(-1)^n/n$ converges, but not absolutely. Such a sequence – convergent but not absolutely – is said to be conditionally convergent.

**4.1 Theorem** *Every absolutely convergent sequence converges.*

*Proof.* If $\sum |a_n|$ converges, then its sequence of partial sums is Cauchy. Now the difference of any two such partial sums is an upper bound for the absolute value of the difference of the corresponding partial sums of $\sum a_n$, by the triangle inequality. Thus the latter sequence is Cauchy, and so $\sum a_n$ converges. □

**Rearrangeing series**

Given a series $\sum a_n$ and a bijection $\sigma : \mathbb{N} \to \mathbb{N}$, there is an associated series $\sum a_{\sigma(n)}$ whose terms are the same, just in a different order. Such a series is called a rearrangement of the original one, and might diverge even if the original one converges. For example the series

$$1 - 1 + \tfrac{1}{2} - \tfrac{1}{2} + \tfrac{1}{2} - \tfrac{1}{2} + \tfrac{1}{3} - \tfrac{1}{3} + \tfrac{1}{3} - \tfrac{1}{3} + \tfrac{1}{3} - \tfrac{1}{3} + \cdots$$

with partial sums $1, 0, \frac{1}{2}, 0, \frac{1}{2}, 0, \frac{1}{3}, 0, \frac{1}{3}, 0, \frac{1}{3}, 0, \ldots$ converges to 0, but its rearrangement

$$1 - 1 + \tfrac{1}{2} + \tfrac{1}{2} - \tfrac{1}{2} - \tfrac{1}{2} + \tfrac{1}{3} + \tfrac{1}{3} + \tfrac{1}{3} - \tfrac{1}{3} - \tfrac{1}{3} - \tfrac{1}{3} + \cdots$$

with partial sums $1, 0, \frac{1}{2}, 1, \frac{1}{2}, 0, \frac{1}{3}, \frac{2}{3}, 1, \frac{2}{3}, \frac{1}{3}, 0, \ldots$ diverges by oscillation. A series is said to converge unconditionally if all its rearrangements converge to the same value.

It is known that a series converges unconditionally if and only if it converges absolutely. Here we prove the "only if" part, and also a surprising fact about conditionally convergent sequences:

**4.2 Riemann's Rearrangement Theorem**

**a)** *Rearrangements of absolutely convergent series will always converge, and to the same limit.*

**b)** *Conditionally convergent series can be rearranged to converge to any prescribed limit, or to diverge to $\pm\infty$, or to diverge by oscillation.*

*Proof.* **a)** Let $\sum a_n$ be an absolutely convergent series converging to $A$, with partial sums $A_k$, and $\sum b_n$ be a rearrangement of $\sum a_n$, with partial sums $B_k$. We must show that $\sum b_n$ converges to $A$. Given $\varepsilon > 0$, choose $k$ so that for all $m > k$,

$$\sum_{n \geq k} |a_n| \ < \ \varepsilon/2 \quad \text{and} \quad |A_m - A| \ < \ \varepsilon/2$$

Now choose $m, \ell$ with $k < \ell < m$ such that $\{a_n \,|\, n \leq k\} \subset \{b_n \,|\, n \leq \ell\} \subset \{a_n \,|\, n \leq m\}$. Then $|B_\ell - A_m| < \varepsilon/2$. Indeed, the first $k$ terms in $A_m$ appear in $B_\ell$, while every term in $B_\ell$ appears in $A_m$. Thus $B_\ell - A_m$ is the sum of *some* of the terms in $\sum_{n>k} a_n$, which is less than $\varepsilon/2$ in absolute value, since $\sum_{n \geq k} |a_n| < \varepsilon/2$. It follows that

$$|B_\ell - A| \ \leq \ |B_\ell - A_m| + |A_m - A| \ \leq \ |B_\ell - A_m| + \varepsilon/2 \ < \ \varepsilon,$$

by the triangle inequality, and so $\sum b_n = A$. The proof of **b)**, which is sketched in Morgan, is left to the reader. ☐

We conclude with two important tests for absolute convergence, the first of computational value, and the second of more theoretical value:

**4.3 Theorem** *Given a series $\sum a_n$, set*

$$\rho \ = \ \lim |a_{n+1}/a_n| \quad \textit{and} \quad r \ = \ \limsup \sqrt[n]{|a_n|}$$

*Note that $\rho$ need not exist, but $r$ always exists.*

**a) (Ratio Test)** *The series converges absolutely if $\rho < 1$, and diverges if $\rho > 1$. If $\rho = 1$ or the limit doesn't exist, the test fails, and the series might converge absolutely, converge conditionally, or diverge.*

**b) (Root Test)** *The series converges absolutely if $r < 1$, and diverges if $r > 1$. If $r = 1$ the test fails, and the series might converge absolutely, converge conditionally, or diverge.*

The proofs of **a)** and **b)** are similar, so we prove **b)**, leaving **a)** for homework. First suppose $r < 1$, and choose $s$ with $r < s < 1$. Then for sufficiently large $n$ we have $\sqrt[n]{|a_n|} < s$, so $|a_n| < s^n$. But then the absolute convergence of the series follows by comparting it to the geometric series $\sum s^n$. If $r > 1$ then the terms in the series do not go to zero, and so it diverges. ☐

## 5. Power Series

Exercises 21 (1, 2, 3, 5)

Just like infinite numerical series, an infinite series of functions is a sum $\sum f_n$ of the functions in an infinite sequence $f_1, f_2, f_3, \ldots$ of functions (all defined on the same domain). It is said to converge pointwise or uniformly according to whether the corresponding sequence of partial sums $F_k = \sum_{n \leq k} f_n$ converges pointwise or uniformly, and to converge absolutely if $\sum |f_n|$ converges (pointwise or uniformly). The most quoted test for convergence in this setting is the

**5.1 Weierstrass $M$-test** *If $\|f_n\| \leq M_n$, where $M_n > 0$ and $\sum M_n$ converges, then $\sum f_n$ converges uniformly.*

*Proof.* For each $x$ in the commond domain of the $f_n$'s, the numerical series $\sum f_n(x)$ converges to some $f(x)$ by comparison with $\sum M_n$. Thus $\sum f_n$ converges pointwise to $f$, and the convergence is uniform since

$$\|f - \textstyle\sum_{n \leq k} f_n\| \;=\; \|\textstyle\sum_{n>k} f_n\| \;\leq\; \textstyle\sum_{n>k} M_n \;\longrightarrow\; 0$$

as $k \to \infty$. □

**Example** $\sum (\sin nx)/n^p$ converges uniformly for any $p > 1$ since $|(\sin nx)/n^p| \leq 1/n^p$ and $\sum 1/n^p$ converges for all $p > 1$.

We now specialize from general series of functions $\sum f_n$ to power series, in which the terms are monomials $f_n(x) = a_n x^n$. Thus a power series is just an infinite polynomial

$$\sum a_n x^n \;=\; a_0 + a_1 x + a_2 x^2 + \cdots .$$

A function that is defined by a convergent power series is called a real analytic function. As we will see below, any real analytic function is $C^\infty$ (i.e. all its derivatives of all orders exist), but not conversely, i.e. there exist $C^\infty$ functions that are not analytic.

We first establish a basic property of power series :

**5.2 Radius of Convergenece** *Every power series $\sum a_n x^n$ has a unique radius of convergence $R \in [0, \infty]$, such that the series converges absolutely for all $|x| < R$ and diverges for all $|x| > R$.*[†] *Furthermore, the series converges uniformly on any compact subset of $(-R, R)$.*

**Examples** The radius of convergence of $\sum x^n/n$

*Proof of* 5.2. Suppose the series converges at $x = r$. If we can show this implies absolute convergence for $|x| < |r|$, then the first statement in 5.1 will follow where $R = \sup\{r \in \mathbb{R} \mid \sum a_n r^n \text{ converges}\}$. But convergence at $x = r \Longrightarrow a_n r^n \to 0$, so $|a_n r^n| \leq 1$ for sufficiently large $n$. Thus for $|x| < |r|$,

$$|a_n x^n| \;=\; |a_n x^n r^n / r^n| \;\leq\; |x/r|^n$$

and so $\sum a_n x^n$ converges absolutely by comparison with the convergent geometric series $\sum |x/r|^n$.

To prove the series converges uniformly on compact subsets of $(-R, R)$, it suffices to show it does so on $[-P, P]$ for any $P < R$ : The series converges at any $Q$ for $P < Q < R$, so $|a_n Q^n| \to 0$, whence $|a_n Q^n| \leq 1$ for sufficiently large $n$. Thus for $|x| \leq P$,

$$|a_n x^n| \;\leq\; |a_n P^n| \;\leq\; |a_n P^n Q^n / Q^n| \;\leq\; (P/Q)^n$$

and so $\sum a_n x^n$ converges uniformly on $[-P, P]$ by the Weierstrass $M$-test, comparing it with the convergent geometric series $\sum (P/Q)^n$. □

From 5.2 and Theorem 1.1b (the integral and limit can be exchanged for uniformly convergent sequences of functions), it follows that ***any power series can be integrated term by term:***

**5.3 Corollary** *If $\sum a_n x^n$ has radius of convergence $R > 0$ and $[a, b] \subset (-R, R)$, then*

$$\int_a^b \sum a_n x^n \, dx \;=\; \sum \int_a^b a_n x^n \, dx \;=\; \sum \frac{a_n x^{n+1}}{n+1} \Bigg|_a^b$$

[†] Thus the interval of convergence of the series is either $(-R, R)$, $[-R, R)$, $(-R, R]$ or $[-R, R]$.

**Example** The geometric series $\sum_{n=0}^{\infty} x^n$ converges iff $x < 1$, so has radius of convergence $R = 1$. Thus for any $b < 1$ we can compute

$$\int_0^b (1 + x + x^2 + \cdots)\, dx \;=\; x + \frac{x^2}{2} + \frac{x^3}{3} + \cdots \Big|_0^b \;=\; b + \frac{b^2}{2} + \frac{b^3}{3} + \cdots$$

But also note that the integrand on the left hand side is a geometric series converging to $1/(1-x)$, and so the left hand side equals $\log 1/(1-b)$. For example if $b = 1/2$ or $2/3$, then we obtain the series expansions

$$\log 2 \;=\; \frac{1}{2} + \frac{1}{8} + \frac{1}{24} + \cdots \qquad \log 3 \;=\; \frac{2}{3} + \frac{4}{18} + \frac{8}{81} + \cdots$$

Now there is a nice formula for the radius of convergence $R$ of any power series :

**5.4 Formulas for $R$** (Hadamard's Root Formula) *The radius of convergence of $\sum a_n x^n$ is*

$$R \;=\; \limsup |a_n|^{-1/n}$$

*which is just $1/\rho$ for the $\rho$ from the root test for $\sum a_n$.*

(Ratio Formula) *A simpler formula, often easier to calculate but not always applicable, is $R = \lim_{n\to\infty} |a_n/a_{n+1}|$ (if that limit exists). This is just $1/r$ for the $r$ from the ratio test for $\sum a_n$.*

*Proof.* Set $\bar{r} = \limsup |a_n|^{-1/n}$ and $\bar{\rho} = \lim |a_n/a_{n+1}|$ (if it exists). For any fixed $x$, set $r = \limsup |a_n x^n|^{1/n}$ and $\rho = \lim |a_{n+1}x^{n+1}/a_n x^n|$ (if it exists). Thus $r = |x|/\bar{r}$ and $\rho = |x|/\bar{\rho}$, so $r < 1 \iff |x| < \bar{r}$ and $\rho < 1 \iff |x| < \bar{\rho}$. By the root (resp. ratio) test, the series converges absolutely at $x$ if $|x| < \bar{r}$ (resp. $|x| < \bar{\rho}$) and diverges if $|x| > \bar{r}$ (resp. $|x| > \bar{\rho}$). By definition, this shows that $R = \bar{r}$, and $R = \bar{\rho}$ (if the latter exists). □

Hadamard's formula has an important consequence with regard to two series associated with a given one $\sum a_n x^n$, its <u>derived</u> <u>series</u> $\sum n a_n x^{n-1}$, and its <u>antiderived</u> series $\sum a_n x^{n+1}/(n+1)$, obtained respectively by differentiating and anti-differentiating the original series term by term.

**5.5 Corollary** *Let $\sum a_n x^n$ be a power series with radius of convergence $R$. Then the derived and antiderived series of $\sum a_n x^n$ also have radius of convergence $R$, and converge in $(-R, R)$ to the derivative $f'$ and antiderivative $\int f$ of the analytic function $f$ represented by $\sum a_n x^n$. Thus any power series can be differentiated and antidifferentiated term by term without changing the radius of convergence, which implies in particular that real analytic functions are $C^\infty$.*

*Proof.* Multiplying the derived series by $x$ yields another series $\sum n a_n x^n$ with the same radius of convergence (since the former converges at $x = a$, say to $b$, if and only if the latter converges at $x = a$ to $ab$) which we compute by Hadamard's formula to be

$$\limsup |n a_n x^n|^{-1/n} \;=\; \limsup |a_n x^n|^{-1/n} \;=\; R$$

since $n^{-1/n} = 1$ (e.g. by L'Hôpital's Rule). Thus the derived series has radius of convergence $R$.

Now the derived series converges to some function $g$ on $(-R, R)$. Fix any $x \in (-R, R)$ and choose $P$ so that $-R < P < x$. It was shown above that $g$ can be integrated term by term, so

$$\int_P^x g \;=\; f(x) - f(P)$$

Taking derivatives of both sides with respect to $x$ (using the Fundamental Theorem of Calculus for the left side) shows that $g(x) = f'(x)$.

Repeating the argument above for the antiderived series in place of the original series shows that both have the same radius of convergence, and that the former converges to $\int f$. □

One consequence of all this is Taylor's explicit formula for the terms in the power series expansion of a real analytic function :

**5.6 Taylor's Theorem** *If* $f(x) = \sum a_n x^n$ *is a real analytic function, then* $a_n = f^{(n)}(0)/n!$, *where* $f^{(n)}$ *denotes the* $n$*th derivative.*

*Proof.* Differentiating $n$ times gives $f^{(n)}(x) = n!a_n + [(n+1)!/1!]a_{n+1}x + [(n+2)!/2!]a_{n+2}x^2 + \cdots$, and so $f^{(n)}(0) = n!a_n$. Now divide by $n!$ □

**6. Fourier Series** Exercises 22 (3, 4, 5) and 24 (1)

DID NOT TEX UP NOTES FROM THE REST OF THE SEMESTER ...