Topology Lecture Notes

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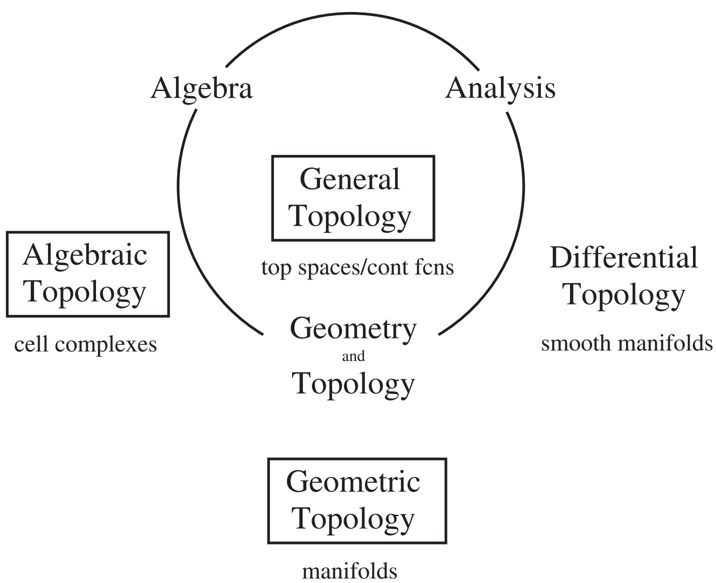


PART I

GENERAL TOPOLOGY

0 Introduction

Euclidean Spaces $\mathbb{R}^n$

- (norm) $\|x\| = (x_1^2 + \cdots + x_n^2)^{1/2}$
- (distance) $d(x,y) = \|x - y\|$ (the standard metric on $\mathbb{R}^n$)
- (continuity) Let $X \subset \mathbb{R}^n$ and $Y \subset \mathbb{R}^k$. Then $f : X \to Y$ is continuous if $\forall\, a \in X$ and $\varepsilon > 0$, there exists $\delta > 0$ such that

$$x \in X,\ d(x,a) < \delta \implies d(f(x), f(a)) < \varepsilon.$$

- (topological equivalence) If $f : X \to Y$ is continuous and has a continuous inverse, then it is called a homeomorphism and we say X and Y are homeomorphic, written $X \cong Y$.

  Examples: $\square \cong \bigcirc$ via $x \mapsto x/\|x\|$, $\bigcirc \not\cong \infty \not\cong \ominus$ (explain why), annulus $\cong$ "twisted" annulus $\not\cong$ Möbius strip (explain why), torus $\cong \square$ w/ identifications $\not\cong$ sphere, etc.

$\boxed{\text{HW \#1}}$ Partition the capital alphabet into homeomorphism classes.

General Topology

Study of topological spaces – a generalization of subsets of $\mathbb{R}^n$ – up to homeomorphism. Important examples include balls and spheres

$$B^n = \{x \in \mathbb{R}^n \mid \|x\| \le 1\} \qquad S^{n-1} = \{x \in \mathbb{R}^n \mid \|x\| = 1\},$$

surfaces (the torus, Klein bottle, Möbius strip ...), 3-dimensional spaces (the 3-sphere S^3, the 3-torus T^3, Poincaré's dodecahedral space, ...) and wilder things (e.g. the long line, the Cantor set, Antoine's necklace, Alexander's horned sphere)

Algebraic Topology

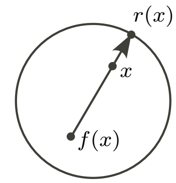
Study of topological properties of spaces using algebra, via "functors"

$$\text{Topology} \xrightarrow{\pi} \text{Algebra}$$

assigning an algebraic object $\pi(X)$ (a group, ring, vector space etc.) to each topological space X, and a homomorphism $f_* : \pi(X) \to \pi(Y)$ to each map $f : X \to Y$ (satisfying $(\mathrm{id}_X)_* = \mathrm{id}_{\pi(X)}$ and $(f \circ g)_* = f_* \circ g_*$)

Example Poincaré's fundamental group $\pi_1(X)$ is a group associated with any space X which measures the "1-dimensional holes" in X. Its elements are represented by loops in X. Facts: $\pi_1(B^2) = 0$, $\pi_1(S^1) = \mathbb{Z}$.

Eight Famous Theorems (proved using topology)



• IN ANALYSIS

① Brouwer Fixed Point Theorem *Any map $B^n \xrightarrow{f} B^n$ has a fixed point.*

Proof (sketch) If f has no fixed points, then the ray from $f(x)$ through x intersects S^{n-1} in a unique point which we call $r(x)$. This defines a continuous function $r : B^n \to S^{n-1}$ which is the identity on S^{n-1}, that is $\mathrm{id}_{S^{n-1}} = r \circ i$ where $i : S^{n-1} \to B^n$ is the inclusion map. This violates the Intermediate Value Theorem when $n = 1$, and the functoriality of π_1 when $n = 2$ (since this would imply $\mathrm{id}_{\mathbb{Z}} = (\mathrm{id}_{S^1})_* = (r \circ i)_* = r_* \circ i_* = 0$). For $n > 1$, one must use a generalization of the fundamental group. In any case one reaches a contradiction.

② Picard Theorem *Any nonconstant analytic function $f : \mathbb{C} \to \mathbb{C}$ assumes all but possibly one value.*

• IN ALGEBRA

③ Fundamental Theorem of Algebra *Any complex polynomial $p : \mathbb{C} \to \mathbb{C}$ has a root.*

④ Nielsen-Schreier Theorem *Any subgroup of a free group is free.*

• IN GEOMETRY/TOPOLOGY

⑤ Hairy Ball Theorem *Any tangent vector field on S^2 must vanish somewhere.* (You can't comb the hair on a porcupine) Note: the result generalizes to S^n for even n.

⑥ Borsuk-Ulam Theorem *Any map $f : S^n \to \mathbb{R}^n$ must identify a pair of antipodal points*

An interpretation for $n = 2$: at any given time, there exists at least one pair of antipodal points on the earth's surface with identical temperature and humidity level.

An application for $n = 3$: The "Ham Sandwich Theorem" asserts that any three bounded convex subsets H, C, B of $\mathbb{R}^3$ (ham, cheese and bread) can be simultaneously bisected with a plane (cut with a knife). To prove this, first identify $\mathbb{R}^3$ with $\mathbb{R}^3 \times \{1\}$ in $\mathbb{R}^4$. Apply Borsuk-Ulam to the map $S^3 \to \mathbb{R}^3$ that sends x to the vector whose three coordinates are the volumes of the parts of H, C and B, respectively, on the same side of $x^\perp$ as x, where $x^\perp$ is the subspace of $\mathbb{R}^4$ perpendicular to x. Then $f(x) = f(-x)$ means that the plane $x^\perp \cap \mathbb{R}^3$ does the job. (Challenge question: Why did we identify $\mathbb{R}^3$ with $\mathbb{R}^3 \times \{1\}$ and not $\mathbb{R}^3 \times \{0\}$?)

⑦ Euler's Theorem *The Euler characteristic $\chi(P)$ $(= v - e + f)$ of the boundary surface S of any convex polyhedral solid in $\mathbb{R}^3$ is equal to 2.*

Proof (sketch) Choose a maximal tree T and its dual graph G in S. Then G is a tree. Now use the observations that $\chi(\text{tree}) := v - e = 1$ (proved by induction) and $\chi(S) = \chi(T) + \chi(G)$.

⑧ Classification of Surfaces *Every closed, connected, orientable surface is homeomorphic to a "multiholed torus"* (draw pictures)

1 Metric Spaces

Definition 1.1 A metric on a set X is a function $d : X \times X \to \mathbb{R}$ satisfying

M1) (positivity) $d(x, y) > 0$ if $x \neq y$, $d(x, x) = 0$

M2) (symmetry) $d(x, y) = d(y, x)$

M3) ($\triangle$ inequality) $d(x, y) + d(y, z) \geq d(x, z)$.

The pair (X, d) is called a metric space; usually suppress d from the notation.

Examples ① $\mathbb{R}^n$ with the standard metric, or any subset with the restricted metric. ② The Hilbert space ℓ^2 = the set of all square summable infinite sequences $x = (x_1, x_2, \dots)$ (meaning $x_1^2 + x_2^2 + \cdots < \infty$) with metric $d(x, y) = \|x - y\|$ where $\|x\| = (x_1^2 + x_2^2 + \cdots)^{1/2}$.

If X is a metric space, then the set

$$B_r(x) \;=\; \{y \in X \mid d(x, y) < r\}$$

where $x \in X$ and $r > 0$ is called an open ball in X of radius r and center x.

Definition 1.2 A function $f : X \to Y$ between metric spaces is continuous if $\forall\, x \in X$ and $\varepsilon > 0$, there exists $\delta > 0$ such that $d(x, y) < \delta \implies d(f(x), f(y)) < \varepsilon$, or equivalently $f(B_\delta(x)) \subset B_\varepsilon(f(x))$.

Key Observation Continuity can be defined without referring to the metric, using the notion of "open sets" defined as follows.

Definition 1.3 A subset U of a metric space is open if every point in U is the center of some open ball lying entirely inside U.

$\boxed{\text{HW \#2}}$ Prove that any open ball in a metric space is open.

Proposition 1.4 *A function $f : X \to Y$ between metric spaces is continuous if and only if $f^{-1}(V)$ is open in X for every open subset V of Y.*

Proof ($\Longrightarrow$) $x \in f^{-1}(V) \implies f(x) \in V \implies B_\varepsilon(f(x)) \subset V$ (for some ε since V is open) $\implies f(B_\delta(x)) \subset B_\varepsilon(f(x))$ (some δ since f is continuous at x) $\implies B_\delta(x) \subset f^{-1}(V)$, and so $f^{-1}(V)$ is open.

($\Longleftarrow$) Given $\varepsilon > 0$, any $x \in X$ lies in the subset $f^{-1}(B_\varepsilon(f(x))$ of X. Since the ball $B_\varepsilon(f(x)$ is open in Y (by HW#2), its preimage $f^{-1}(B_\varepsilon(f(x))$ is open in X, and so contains $B_\delta(x)$ for some $\delta > 0$. This means $f(B_\delta(x)) \subset B_\varepsilon(f(x))$, so f is continuous. $\square$

Remarks 1.5 a) The intersection of *finitely* many open sets in a metric space X is open (at each point in the intersection, take the min of the radii needed for each of the sets), as is the union of *arbitrarily* many open sets. So are X and $\varnothing$. It follows that $U \subset X$ is open $\iff U$ is the union of a (possibly empty) collection of open balls ($\Longleftarrow$ uses HW #2).

b) Different metrics may give rise to the same open sets. For example, for each real number $p \geq 1$, the norms

$$\|x\|_p = (\sum |x_i|^p)^{1/p} \quad \text{(for any real } p \geq 1) \qquad \text{and} \qquad \|x\|_\infty = \max |x_i|$$

induce metrics d_p and d_∞ on $\mathbb{R}^n$, *all* with the same open sets (d_2 is the standard metric, and d_∞ is called the sup metric). To prove this, it suffices to show that for any $p, q \in [1, \infty]$ and any ball B for d_p, there is a concentric ball B' for d_q such that $B' \subset B$.

$\boxed{\text{HW \#3}}$ Describe the open balls in the metrics d_1, d_2, d_3 and d_∞ on $\mathbb{R}^2$. Prove using pictures that these metrics have the same open sets. (Hint: it suffices to show each ball in one metric centered at $a \in]br^2$ contains a ball centered at a in the other metric.) Explain why $d_{1/2}$, defined in the same way, is not a metric. (Hint: find *one* example in which one of the axioms **M1) 2)** or **3)** fails.)

$\boxed{\text{HW \#4}}$ For any metric d on a set X, define $\hat{d} : X \times X \to \mathbb{R}$ by

$$\hat{d}(x, y) = \frac{d(x, y)}{1 + d(x, y)}.$$

Show that $\hat{d}$ is also a metric on X, and that d and $\hat{d}$ have the same open sets. Note that X is "bounded" with respect to $\hat{d}$ (meaning $\exists M$ such that $\hat{d}(x, y) < M$ for all $x, y \in X$) although it need not be bounded with respect to d.

Remark 1.6 (Heine's definition) Continuity can also be defined in terms of convergent sequences, as follows: A function $f : X \to Y$ is continuous iff $\forall\, x \in X$,

$$x_n \to x \implies f(x_n) \to f(x).$$

Here $x_n \to x$ means $\forall \varepsilon > 0$, $\exists N$ such that $x_n \in B_\varepsilon(x)$ for all $n > N$. This is often the way continuity is used in analysis courses. It is a worthwhile exercise to prove that this is equivalent to the ε-δ definition.

2 Topological Spaces

Definition 2.1 A topology on a set X is a collection $\mathcal{O}$ of subsets of X,[†] called open sets, that satisfy (cf. Remark 1.5a)

O1) the intersection of any finite number of open sets is open

O2) the union of any (possibly infinite) number of open sets is open

O3) X and $\varnothing$ are open.

The pair $(X, \mathcal{O})$ is called a topological space (or just space). We usually suppress $\mathcal{O}$ from the notation, but don't forget that the notion of "openness" depends on the topology; a subset of X can be open in one topology on the set X and not open in another!

A function $f \colon X \to Y$ between spaces is continuous if $f^{-1}(V)$ is open in X for every open V in Y. Continuous functions are also called maps. Observe that the composition

$$X \xrightarrow{f} Y \xrightarrow{g} Z$$

of maps is a map: W open in $Z \Longrightarrow (g \circ f)^{-1}(W) = f^{-1}(g^{-1}(W))$ open in X. A map which has a continuous inverse is called a homeomorphism. Two spaces X and Y are said to be homeomorphic, written $X \cong Y$, if there exists a homeomorphism $X \to Y$.

Examples of Topological Spaces

① Any set X with the trivial topology $\{X, \varnothing\}$, or the discrete topology $\mathcal{P}(X)$ (= the power set of X, i.e. *all* subsets of X are open in the discrete topology). Note that any function with domain a discrete space, or codomain a trivial space, is continuous.

② Any set with the cofinite topology $\{U \mid U = \varnothing \text{ or } X - U \text{ is finite}\}$. (Verify the axioms)

③ Any metric space (X, d) with its associated metric topology $\mathcal{O}_d$, consisting of the open sets defined as above using open balls (see Remark 1.4a). For example the metric topology for the standard metric on $\mathbb{R}^n$ (or any of the metrics d_p defined above) is called the "standard topology" on $\mathbb{R}^n$. A space whose topology is the metric topology for some metric is said to be metrizable. For example any discrete space is metrizable, with $d(x, y) := 1$ for all $x \neq y$. Not all spaces are metrizable, however, as we will see below!

④ Finite sets have lots of different topologies. For example any set with 3 elements has 29 different topologies, which fall into 9 homeomophism classes. (Can you find them all? Draw pictures) In general, let t_n denote the number of distinct topologies on an n element set X, and let h_n denote the number of homeomorphism classes of the associated spaces. Then the first seven values of t_n and h_n are, respectively, $1, 4, 29, 355, 6942, 209527, 9535241$ and $1, 3, 9, 33, 139, 718, 4535$, but no general formulas are known!

[†] in other words $\mathcal{O}$ is a subset of the power set $\mathcal{P}(X)$ = the set of all subsets of X

Subspaces

Any subset A of a space X has a natural subspace or relative topology consisting of $\{U \cap A \mid U \text{ is open in } X\}$. With this topology, A is called a subspace of X. The inclusion map $i\colon A \to X$ of any subspace, defined by $i(a) = a$, is continuous, since $i^{-1}(U) = U \cap A$. In fact this shows that the subspace topology is the smallest topology on A for which i is continuous.

Sums and Products

If X and Y are spaces, then so is their disjoint union $X \sqcup Y$ with the sum topology (consisting of all sets of the form $U \sqcup V$ for open $U \subset X,\ V \subset Y$) and their cartesian product $X \times Y$ with the product topology (consisting of all sets W for which each $x \in W$ lies in an open box – meaning any set of the form $U \times V$ for open $U \subset X,\ V \subset Y$ – in $W^{\dagger}$). The former is called the sum (or disjoint union), and latter the product, of X and Y. (draw pictures)

$\boxed{\text{HW \#5}}$ Show that the product topology on $X \times Y$ (as defined above) is indeed a topology.

The product $X \times Y$ has "natural projections" $X \xleftarrow{p_1} X \times Y \xrightarrow{p_2} Y$ defined by $p_1(x, y) = x$ and $p_2(x, y) = y$. Likewise the sum $X \sqcup Y$ has "natural inclusions" $X \xrightarrow{i_1} X \sqcup Y \xleftarrow{i_2} Y$ defined by $i_1(x) = x$ and $i_2(y) = y$. These are all continuous and satisfy the following:

Universal Properties 2.2 If W is a space equipped with maps $X \xleftarrow{q_1} W \xrightarrow{q_2} Y$ (call this gizmo a "pre-product") then $\exists!$ map $W \xrightarrow{g} X \times Y$ such that $q_1 = p_1 g$ and $q_2 = p_2 g$, i.e. the following diagram commutes:

$$
\begin{tikzcd}
 & W \arrow[dl, "q_1"'] \arrow[d, dashed, "g"] \arrow[dr, "q_2"] & \\
X & X \times Y \arrow[l, "p_1"] \arrow[r, "p_2"'] & Y
\end{tikzcd}
$$

If Z is a space equipped with maps $X \xrightarrow{j_1} Z \xleftarrow{j_2} Y$ (call this a "pre-sum") then $\exists!$ map $X \sqcup Y \xrightarrow{h} Z$ such that $j_1 = h i_1$ and $j_2 = h i_2$, i.e. the following diagram commutes:

$$
\begin{tikzcd}
X \arrow[r, "i_1"] \arrow[dr, "j_1"'] & X \sqcup Y \arrow[d, dashed, "h"] & Y \arrow[l, "i_2"'] \arrow[dl, "j_2"] \\
 & Z &
\end{tikzcd}
$$

Thus given X and Y, their product is an "initial object" (a.k.a. a "universally repelling object") in the category of all pre-products, and their sum is a "teminal object" (a.k.a. "universally attracting object") in the category of all pre-sums. In a general category, such constructs are called "products" and "coproducts".

The proofs are left as exercises. The key ingredient in the case of the product is:

†or equivalently, the product topology consists of all sets that are unions of open boxes.

$\boxed{\text{HW \#6}}$ Prove that any function $W \xrightarrow{f} X \times Y$, whose compositions $p_1 f$, $p_2 f$ with the projections onto the factors are continuous, is continuous.

Bases (used to efficiently define and compare topologies on a set X)

For *any* $\mathcal{S} \subset \mathcal{P}(X)$, there is a unique smallest topology $\mathcal{O}(\mathcal{S})$ on X that contains $\mathcal{S}$, namely the collection of all unions of sets that are themselves intersections of finitely many sets in $\mathcal{S}$; this is a topology by deMorgan's laws.[†] We say $\mathcal{S}$ generates $\mathcal{O}(\mathcal{S})$. If the sets in $\mathcal{O}(\mathcal{S})$ can all be expressed simply as unions of sets in $\mathcal{S}$, then we call $\mathcal{S}$ a basis for $\mathcal{O}(\mathcal{S})$. In other words, a basis for a topology is just a collection of some of its open sets – designated as "basic" – such that any open set can be written as a union of basic open sets.

Remark If X and Y are spaces and $\mathcal{S}$ is a basis for the topology on Y, then a function $f : X \to Y$ is continuous if and only if $f^{-1}(V)$ is open in X for every $V \in \mathcal{S}$. This follows from deMorgan's laws.

Examples of bases ① The 1-element subsets form a basis for the discrete topology.

② The open balls in a metric space form a basis for the metric topology.

③ The infinite strips $J \times \mathbb{R}$ and $\mathbb{R} \times J$, for all open intervals J, generate the standard topology on $\mathbb{R}^2$. However they do not form a basis for this (or any) topology on $\mathbb{R}^2$ (since, for example, the intersection $J \times \mathbb{R} \cap \mathbb{R} \times J = J \times J$ which is not a union of strips).

This last example shows that not every $\mathcal{S} \subset \mathcal{P}(X)$ is the basis for a topology on X, but it is easy to characterize those that are (the proof is left as an exercise):

Lemma 2.3 (characterization of bases) $\mathcal{S} \subset \mathcal{P}(X)$ *is a basis for a topology on X if and only if* (a) *$\mathcal{S}$ covers X* (*meaning X is the union of the sets in $\mathcal{S}$*), *and* (b) *the intersection of any two sets in $\mathcal{S}$ is a union of sets in $\mathcal{S}$. Furthermore, if $\mathcal{S}$ and $\mathcal{S}'$ are bases for topologies $\mathcal{O}$ and $\mathcal{O}'$, then $\mathcal{O} \subset \mathcal{O}' \iff$ every set in $\mathcal{O}$ is a union of sets in $\mathcal{O}'$.*

If a space X has a countable basis, then it is said to be second countable. For example, $\mathbb{R}^n$ is second countable: the open balls whose radii and center coordinates are rational form a countable basis. This is a very useful condition to impose, included in the definition of a "manifold" (below) and related to metrization results.

Hausdorff Spaces

A space X is Hausdorff if any two points $x, y \in X$ have disjoint neighborhoods in X. Here a neighborhood of a point in X, or more generally of a subset A of X, is by definition any subset of X (not necessarily open) that contains an open set containing A; thus it is equivalent to require that x and y lie in disjoint open sets in X.

For example discrete spaces are Hausdorff, as are metric spaces (since any two points $x \neq y$ lie in the disjoint balls $B_{r/2}(x)$ and $B_{r/2}(y)$, where $r = d(x, y)$). In contrast, trivial spaces with more than one element and infinite cofinite spaces are not Hausdorff, hence not metrizable (exercise).

[†] In particular, use the fact that $(\cup_i A_i) \cap (\cup_j B_j) = \cup_{i,j}(A_i \cap B_j)$.

The Hausdorff condition is *extremely* useful. It is clearly a topological property (i.e. any space homeomorphic to a Hausdorff space is Hausdorff) and "hereditary", i.e. inherited by subspaces (exercise). It also behaves well with respect to sums and products:

$\boxed{\text{HW \#7}}$ Prove that two non-empty spaces X and Y are Hausdorff $\Longleftrightarrow$ their sum $X \sqcup Y$ is Hausdorff $\Longleftrightarrow$ their product $X \times Y$ is Hausdorff.

The Hausdorff condition implies that limits of sequences are unique: A sequence $x_1, x_2, \dots$ of points in a space X is said to converge to a point $x \in X$, written $\lim x_n = x$, if for every neighborhood N of x, the sequence is eventually in N (meaning $x_n \in N$ for all sufficiently large n). In a Hausdorff space, $a = \lim x_n = b \Longrightarrow a = b$; the proof is an exercise for the reader. This need not be the case in a space that is not Hausdorff. For example in any space with the trivial topology, every sequence converges to every point.

Manifolds

The Hausdorff condition is generally included in the definition of "manifolds", the primary objects of study for topologists:

Definition 2.4 A manifold is a locally Euclidean, second countable Hausdorff space M. Here "locally Euclidean" means that there is an integer $n \geq 0$ such that each point in M lies in an open set (called a "chart") homeomorphic to an open set in $\mathbb{R}^n$; n is called the dimension of the manifold, denoted $\dim(M)$, and we refer to M as an n-manifold.

Examples of 1-manifolds are $\mathbb{R}$ and the circle S^1, and of 2-manifolds are $\mathbb{R}^2$, any open subset of $\mathbb{R}^2$, the sphere S^2, the torus $T^2 \cong S^1 \times S^1$, etc. The study of 3 and 4-manifolds is a very active area of current research, and in some sense the richest part of the study of manifolds in general; in fact some of the most interesting phenomena in low dimensions, and especially in dimension 4, disappear in higher dimensions!

Special Subsets

$X =$ space, $A \subseteq X$:

- A is closed if its complement $X - A$ is open. Finite unions and arbitrary intersections of closed sets are closed, as are X and $\emptyset$. This follows from deMorgan's law: The complement of a union/intersection of subsets of a set X is the intersection/union of their complements.

Warning: There are subsets of X that are both open and closed (e.g. X and $\emptyset$) and there may be subsets that are neither (give examples in $\mathbb{R}^2$).

Remark A function $f : X \to Y$ between spaces is continuous if and only if the preimage $f^{-1}(C)$ of every closed subset C of Y is a closed subset of X (since $X - f^{-1}(C) = f^{-1}(Y - C)$).

- In general, the closure of A is the set

$$\overline{\mathrm{A}} \;=\; \cap\,\{\text{closed } C \supseteq A\} \;=\; \text{the } \textit{smallest} \text{ closed set containing } A.$$

Clearly A is closed $\Longleftrightarrow A = \overline{\mathrm{A}}$. If X is a metric space, then $\overline{\mathrm{A}}$ can equivalently be defined as the set of limit points of sequences in A, but this is not true in general.

The interior of A is the set

$$A^\circ \;=\; \cup\,\{\text{open } U \subseteq A\} \;=\; \text{the } \textit{largest} \text{ open subset of } A.$$

Clearly A is open $\iff A = A^\circ$. Note that A° can also be described as the set of all interior points of A, which by definition are the points which have a neighborhood entirely inside A.

- There are three other important sets associated with A, its

exterior $A^\times \;=\; \{x \in X \mid \text{some neighborhood of } x \text{ is disjoint from } A\}$
boundary $\partial A \;=\; \{x \in X \mid \text{every neighborhood of } x \text{ intersects } A \text{ and } X - A\}$
limit set $A' \;=\; \{x \in X \mid \text{every neighborhood of } x \text{ contains points in } A \textit{ distinct} \text{ from } x\}$

The points in $A^\times$, ∂A, and A' are called, respectively, exterior, boundary, and limit (or cluster or accumulation) points of A in X. Clearly X is the disjoint union

$$X = A^\circ \;\sqcup\; \partial A \;\sqcup\; A^\times.$$

Also note that $A^\times = (X - A)^\circ$, and so $A^\times$ is open. It follows that ∂A is closed, and that

$$\overline{A} = A^\circ \cup \partial A = A \cup \partial A.$$

$\boxed{\text{HW \#8}}$ For any subset A of a space X, show (using the definitions above) that $\overline{A} = A \cup A'$, and that A is closed $\iff A \supseteq A'$.

If $\overline{A} = X$, i.e. every open set in X intersects A, say A is dense in X. For example, $\mathbb{Q}$ is dense in $\mathbb{R}$. If a space has a countable dense subset, then it is said to be separable.

$\boxed{\text{HW \#9}}$ Show that every second countable space is separable, and that every separable *metric space* is second countable.

Aside

Baire Category Theorem (useful in analysis) *If X is a complete*[†] *metric space and $U_1, U_2, \dots$ are open dense subsets of X, then $U = U_1 \cap U_2 \cap \cdots$ is dense in X.*

Proof It suffices to show V open $\implies V \cap U \neq \varnothing$. Inductively construct balls $B_1 \supset B_2 \supset \cdots$ in V with $B_n \subset U_1 \cap \cdots \cap U_n$ for each n. Completeness $\implies B_1 \cap B_2 \cap \cdots$ is a nonempty subset of $V \cap U$. □

Application $\exists$ continuous, nowhere differentiable functions $I \to \mathbb{R}$, where $I = [0, 1]$.

Proof Apply Baire to $X = \{\text{continuous functions } I \xrightarrow{f} \mathbb{R}\}$ with the "sup metric"

$$d(f, g) = \max_{t \in I} |f(t) - g(t)|$$

and with $U_n = \{f : \forall t \in I, \exists\, h \text{ with } |f(t+h) - f(t)| > nh\}$. □

[†] meaning that any Cauchy sequence in X converges to a point in X

Piecing maps together

Piecewise continuous functions (e.g. functions defined by more than one formula) need not be continuous, but under certain mild conditions on the "pieces", they are:

Gluing Lemma 2.5 *If $f : X \to Y$ is a function between spaces such that $f|A$ and $f|B$ are continuous, where A and B are closed subsets of X with $A \cup B = X$, then f is continuous. The same conclusion holds if A and B are both open in X.*

Proof If C is closed in Y, then $f^{-1}(C) = (f|A)^{-1}(C) \cup (f|B)^{-1}(C)$ is the union of two closed subsets of X, and so closed in X. Indeed $(f|A)^{-1}(C)$ is closed in A, i.e. the intersection of a closed set in X with A, and so is closed in X since A is closed in X; similarly for $(f|B)^{-1}(C)$. Thus f is continuous. The argument in the open case is analogous. $\square$

3 Separation and Metrization

We first discuss the separation axioms of Alexandroff-Hopf (1945)which generalize the Hausdorff condition. They are useful tools and lead to metrization results, the most famous of which (due to Urysohn) is proved below.

Let A, B be disjoint subsets of a space X. Say that A can be separated from B if A has a nbd disjoint from B, and that A and B can be separated in X if they have disjoint nbds.

Separation Axioms A top space X is

- T_0 if at least one pt of each pair of pts can be separated from the other
- T_1 if either pt of each pair of pts can be separated from the other
- T_2 if each pair of pts can be separated in X (= the Hausdorff condition)
- T_3 if each closed set and pt not in it can be separated in X
- T_4 if each pair of disjoint closed sets can be separated in X

(Draw pictures) Further terminology: $\boxed{\text{regular}} = T_1 + T_3$, $\boxed{\text{normal}} = T_1 + T_4$.

Facts about T_i-spaces : ① A space X is $T_1 \iff$ all points in X are closed (exercise).

② normal $\Longrightarrow$ regular $\Longrightarrow$ Hausdorff $\Longrightarrow T_1 \Longrightarrow T_0$ (easy consequence of ①).

③ All reverse implications in ② fail (see Steen-Seebach: "Counterexamples in Topology").

④ While regularity and the Hausdorff conditions are hereditary (inherited by subspaces), normality is not, although it is weakly hereditary (inherited by closed subsets) – easy exercise.

$\boxed{\text{HW \#10}}$ Prove that any metric space (X, d) is normal.[†]

Not all normal spaces are metrizable (see e.g. Steen and Seebach). However, all second countable ones are, indeed:

Urysohn Metrization Theorem 3.1 (1924) *Any second countable regular topological space is metrizable.*

Tychonoff's Lemma 3.2 *Second countable regular spaces are normal.*

Proof. Let X be second countable, $A, B \subset X$ closed, $A \cap B = \emptyset$. Each $a \in A$ has an open nbd U_a with $\overline{U}_a \cap B = \emptyset$. The cover $\{U_a\}$ of A has a *countable* subcover. (Pf: $a \in$ basic $B_a \subset U_a$. $\{B_a\}$ countable, so $= \{B_i\}$ for $i = 1, 2, \ldots$. Choose $U_i \supset B_i$.) Similarly have a

[†]Hint: For A, B disjoint closed subsets of X, explain why there is an open ball around each point in A that is disjoint from B, and an open ball around each point in B that is disjoint from A. Now use these balls to construct your separation of A and B.

countable cover $\{V_i\}$ of B with $\overline{\mathrm{V}}_i \cap A = \emptyset$. Set $U_i' = U_i - \cup_{j<i}\overline{V}_j$ and $V_i' = V_i - \cup_{j\le i}\overline{U}_j$ (note that $\overline{S \cup T} = \overline{S} \cup \overline{T}$, but not for infinite unions). Then have disjoint open neighborhoods:

$$\begin{aligned} U &= U_1 \cup (U_2 - \overline{\mathrm{V}}_1) \cup (U_3 - \overline{V_1 \cup V_2}) \cup \cdots \;=\; \cup U_i' \\ V &= (V_1 - \overline{\mathrm{U}}_1) \cup (V_2 - \overline{U_1 \cup U_2}) \cup \cdots \;=\; \cup V_i' \end{aligned}$$

of A and B respectively (draw picture).† □

<u>Urysohn's Lemma 3.3</u> *A, B disjoint closed subsets of a normal space $X \Longrightarrow \exists$ continuous $f : X \to I$ with $f(A) = 0$, $f(B) = 1$.*

Proof. Inductively define C_r for every diadic rat'l $r \in (0,1)$ so that $X - C_r =$ disj union of two open sets U_r and V_r w/ $\cup_{s<r} C_s \subset U_r$ and $\cup_{s>r} C_s \subset V_r$ (picture). Define $f : X \to I$ by

$$f(x) = \inf\{r : x \in U_r\}$$

(where $\inf \emptyset = 1$). Note that I has basis of (r,s) with r,s diadic. Since $f^{-1}(r,s) = V_r \cap U_s$ is open, f is continuous. □

Proof of metrization theorem. Let X be second countable and regular, and therefore normal by Tychonoff. Idea: Show X embeds in ℓ^2 ($\Longrightarrow X$ is metrizable since ℓ^2 is).

<u>Definition 3.4</u> A map $f : X \to Y$ is an <u>embedding</u> if it is a homeomorphism onto its image, i.e. it is 1-1, continuous and maps open sets in X to open sets in $f(X)$. Indicate this by writing $f : X \hookrightarrow Y$.

Choose a countable basis $U_1, U_2, \ldots$ for X. For every pair i,j with $\overline{\mathrm{U}}_i \subset U_j$, define $f_{ij} : X \to I$ with $f_{ij}(\overline{\mathrm{U}}_i) = 0$, $f(X - U_j) = 1$ (by Urysohn). Reindex $i,j \leftrightarrow n$. Define $f : X \to \ell^2$ by

$$f(x) = (f_1(x), f_2(x)/2, f_3(x)/3, \cdots).$$

This is in ℓ^2 since $\sum (f_n(x)/n)^2 < \sum 1/n^2 < \infty$.

$\boxed{\text{Claim}}$ f is an embedding:

<u>f is $1-1$</u> Given $x \ne y$ in X, $\exists \overline{\mathrm{U}}_i \subset U_j$ with $x \in U_i$, $y \notin U_j$ (by regularity) $\Longrightarrow$ associated coordinate of $f(x)$ and $f(y)$ differ ($= 0, 1/n$ resp.)

<u>f is continuous</u> Given $x \in X$, $\varepsilon > 0$, it suffices to show $\exists$ (open) nbd U of x such that for all $y \in U$

$$d(f(x), f(y)) \;=\; (\sum d_n(x,y))^{1/2} \;<\; \varepsilon$$

where $d_n(x,y) := (f_n(x)/n - f_n(y)/n)^2$. To do this, choose N ao that $\sum_{n>N} d_n(x,y) < \varepsilon^2/2$ for all $y \in X$; such an N exists since $\sum 1/n^2$ converges and $d_n(x,y) \le 1/n^2$. Next choose U so that $d_n(x,y) < \varepsilon^2/2N$ for all $y \in U$ and $n \le N$ (by continuity of f_n). This implies that

†Clearly $A \subset U$ (any $a \in A$ lies in some U_i but in no $\overline{V}_j$, so lies in U_i'). Similarly $B \subset V$ Finally $U \cap V$ is empty: If not, then $\exists x \in U_i' \cap V_k'$ for some i,k. If $i > k$ then $x \notin \overline{V}_k$ but is in V_k (contradiction) while if $i \le k$ then $x \in U_i$ but is not in $\overline{U}_i$ (again a contradiction).

$\sum_{n \leq N} d_n(x,y) < \varepsilon^2/2$ for $y \in U$. Therefore the whole sum $\sum d_n(x,y) < \varepsilon^2$ for all $y \in U$, so $d(f(x), f(y)) < \varepsilon$.

f is open ISTS U open and $x \in U \implies f(U)$ contains $B_\varepsilon(f(x)) \cap f(X)$ for some $\varepsilon > 0$. As above, the regularity of X shows that $\exists$ basic U_i, U_j with $x \in \overline{U}_i \subset U_j \subset U$. If n corresponds to ij then $d(f(x), f(y)) > 1/n$ for $y \notin U_j$, and so $f(U)$ contains $B_{1/n}(f(x)) \cap f(X)$. $\square$

Another application of Urysohn's Lemma:

Tietze Extension Theorem 3.5 *C closed subset of normal X, $f : C \to [-1,1]$ continuous $\implies \exists$ continuous $F : X \to [-1,1]$ extending f, i.e. $F|C = f$.* (Exercise: Find a counterexample when C is not closed)

Lemma *Given $h : C \to [-m,m]$, $\exists\, g : X \to [-\frac{m}{3}, \frac{m}{3}]$ with $|h - g| \leq \frac{2m}{3}$ on C.*

Proof. Set $A = h^{-1}[-m, -\frac{m}{3}]$ and $B = h^{-1}[\frac{m}{3}, m]$, and apply Urysohn's Lemma to get $g : X \to [-\frac{m}{3}, \frac{m}{3}]$ with $g(A) = -\frac{m}{3}$ and $g(B) = \frac{m}{3}$. $\square$

Proof. (of Tietze) Apply lemma to $h = f$ to find g_1 with $|g_1| \leq \frac{1}{3}$, $|f - g_1| \leq \frac{2}{3}$ on C, then to $h = f - g_1$ to get g_2 with $|g_1| \leq (\frac{1}{3})^2$, $|f - (g_0 + g_1)| \leq (\frac{2}{3})^2$ on C, etc. In other words, induct to find g_n with

$$\text{ⓐ}\quad |g_n| \leq (\tfrac{1}{3})^n\,, \qquad \text{ⓑ}\quad |f - \textstyle\sum_{i=1}^n g_i| \leq (\tfrac{2}{3})^n \quad \text{on } C.$$

Set $F = \sum g_n$. F is cont by ⓐ (Weierstrass M-test) and $F|C = f$ by ⓑ. $\square$

General Extension Problem

$$\begin{tikzcd} X \arrow[dr, "\exists ?\, F"] & \\ A \arrow[u, "i", hook] \arrow[r, "f"'] & Y \end{tikzcd}$$

No general solution is known, but many special cases are understood (e.g. A closed subset of normal X, and $Y = I$ or $\mathbb{R}$.) Method of algebraic topology:

nonexistence apply appropriate functor H : Topology $\to$ Algebra

$$\begin{tikzcd} H(X) \arrow[dr, "\nexists\, G"] & \\ H(A) \arrow[u, "H(i)"] \arrow[r, "H(f)"'] & H(Y) \end{tikzcd}$$

(Give example of Brouwer Fixed Point Theorem again: $f = \mathrm{id} : S^1 \to S^1$, and $X = B^2$).

existence obstruction theory (blend of homotopy and cohomology theory)

$\boxed{\text{HW \#11}}$ Show that any manifold M is metrizable. You may use without proof the fact that any closed ball in a chart in M is closed in M as well (an easy consequence of compactness, yet to come).

4 Connected Spaces

We discuss two basic, closely related notions: connected and path-connected spaces X. Intuitively the first means that X consists of a single "piece", and the second (which turns out to be slightly more restrictive) means that there exist "paths" between any two points in X.

Connectedness

A clopen subset of a space X is a subset that is both closed and open in X. For example $\varnothing$ and X are clopen in X; these are called the trivial clopen subsets.

Definition 4.1 X is connected if it has no nontrivial clopen subsets.

Equivalent formulations X is connected if and only if either of the following conditions holds ① X is *not* the disjoint union of two non-empty open subsets, or ② every map from X to a discrete space is constant. (Exercise: Show that these definitions are all equivalent)

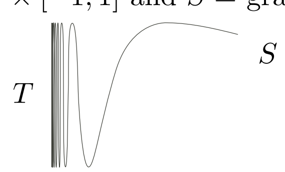
Connectedness is clearly a topological property, meaning any space homeomorphic to a connected space is connected. But it is not not a hereditary property, meaning subspaces of connected spaces need not be connected, as easily seen from the following characterization of the connected subspaces of $\mathbb{R}$:

Theorem 4.2 *A subspace J of $\mathbb{R}$ is connected iff it is convex (i.e. an interval). In particular $\mathbb{R}$ and the unit interval $I = [0, 1]$ are connected.*

Proof. ($\Longrightarrow$) Sps J connected but not convex. Then $\exists\ a < b < c$ with $a, c \in J$ and $b \notin J$. But then $J \cap [b, \infty) = J \cap (b, \infty)$ is a nontrivial clopen subset of $J \Longrightarrow\!\Longleftarrow$.

($\Longleftarrow$) If J is convex but not connected, then it has a nontrivial clopen subset A. Choose $a \in A$, $c \in J - A$. WLOG $a < c$, so $[a, c] \subset J$ (by convexity). Now consider $b = \operatorname{lub}(A \cap [a, c])$. If $b \in A$, then $b \in$ some $[b, b + \varepsilon) \subset A$ (since A is open) $\Longrightarrow b \neq$ an upper bound for $A \cap [a, c] \Longrightarrow\!\Longleftarrow$. Sim'ly $b \in J - A \Longrightarrow$ some $(b - \varepsilon, b] \subset J - A \Longrightarrow\!\Longleftarrow$ minimality of b. $\square$

Remark Connected subspaces of $\mathbb{R}^2$ can be quite wild, e.g. the topologist's sin curve $TS = T \cup S \subset \mathbb{R}^2$, where $T = \{0\} \times [-1, 1]$ and $S =$ graph of $f(x) = \sin(1/x)$ on $(0, 1]$.



To see that TS is connected, need (part of) the following:

Proposition 4.3 (Properties of connected spaces)

(*a*) *X connected, $f : X \to Y$ continuous $\Longrightarrow f(X)$ connected* (Corollary: Int Value Thm)

(*b*) *$X = \cup X_i$, X_i connected, $X_i \cap X_j$ nonempty for all $i, j \Longrightarrow X$ connected*

(*c*) *$S \times T$ is connected $\Longleftrightarrow S$ and T are connected* (*d*) *$A \subset B \subset \overline{A} \subset X$ with A connected $\Longrightarrow$ B is connected (in particular $\overline{A}$ is connected).*

Proof (*a*) The preimage of any nontrivial clopen set in Y is a nontrivial clopen set in X.

For (*b*) observe that any nonempty clopen subset A of X intersects each X_i in a clopen set ($= \varnothing$ or X_i) so A is the union of some of the X_i's. Since each pair of X_i's intersect, A is in fact the union of all of them, i.e. $A = X$.

($c \Longrightarrow$) follows from (*a*) since the natural projections are continuous. For ($c \Longleftarrow$) consider $X_{st} = S \times t \cup s \times T$, for $(s,t) \in S \times T$ (draw picture). (Here we write $s \times T$ for $\{s\} \times T$, etc., by abuse of notation.) Since $S \times t \cong S$ and $s \times T \cong T$ (exercise), and $S \times t \cap s \times T = \{(s,t)\} \neq \varnothing$, it follows from (*b*) that X_{st} is connected. Then $X_{st} \cap X_{s't'} \supset \{(s,t'),(s',t)\} \neq \varnothing$ and $S \times T = \cup X_{st}$, so $S \times T$ is connected by (*b*). We leave (*d*) for HW. $\square$

$\boxed{\text{HW \#12}}$ Prove Proposition 4.3(d)

$\boxed{\text{HW \#13}}$ Prove that the topologist's sin curve is connected. (Hint: use Proposition 4.3)

Components

A component of a space X is a *maximal* connected subspace C of X (meaning that $C \subsetneq D \subset X \Longrightarrow D$ not connected).

Corollary 4.4 *Any space is the disjoint union of its components, each of which is closed.* (These are the "pieces" of the space.)

Proof First note that each $x \in X$ lies in a component, namely $C_x =$ union of all connected $A \subset X$ containing x (this set is connected by 4.3(b) and maximal by construction). Also, distinct components C, D are disjoint: $C \cap D \neq \varnothing \Longrightarrow C \cup D$ connected $\Longrightarrow C = C \cup D = D$ (by maximality). Components are closed by 4.3(d). $\square$

Remark Components need not be open (e.g. the components of $\mathbb{Q}$, as a subspace of $\mathbb{R}$, are points†) but are if there are only finitely many (explain why).

Path-Connectedness and Path-Components

Let x, y be points in a space X. A path in X from x to y is a (continuous) map $\alpha : I \to X$ with $\alpha(0) = x$ and $\alpha(1) = y$. (Recall $I = [0,1]$.)

Definition 4.5 X is path-connected if any two points in X can be joined by a path.

Clearly intervals are path-connected (use the path $\alpha(t) = x + t(y - x)$ to join x to y). Also the analogues of properties 4.3(a)(b)(c), but not (d), e.g. the topologists sin curve $TS = \overline{S}$ is not path-connected although S is:

$\boxed{\text{HW \#14}}$ Prove that the topologists sin curve is not path-connected. (This is tricky)

†such a space is called totally disconnected

Proposition 4.6 (Properties of path-connected spaces)

(a) *The continuous image of a path connected space is path-connected.*

(b) *The union of pairwise intersecting path-connected spaces is path-connected.*

(c) *The product of two spaces is path-connected iff the factors are path-connected.*

Proof (a) If $f : X \to Y$ is continuous, then any $f(x)$ and $f(y)$ in $f(X)$ can be joined by the path $f \circ \alpha$ for any path α in X joining x to y. For (b), it suffices to show that if X and Y are path-connected with $X \cap Y \neq \varnothing$, then we can join any $x \in X$ and $y \in Y$ by a path in $X \cup Y$. So choose $z \in X \cap Y$, and paths α in X from x to z and β in Y from z to y. Then the composite path $\alpha \cdot \beta$ from x to y defined by

$$\alpha \cdot \beta(t) = \begin{cases} \alpha(2t) & \text{for } t \leq 1/2 \\ \beta(2t-1) & \text{for } t \geq 1/2 \end{cases}$$

(which is continuous by the gluing lemma above) is the desired path. Part (c) follows exactly as in the proof of Proposition 4.3(c). □

As for components, the path-components of a space are its maximal path-connected subspaces. Readily deduce the analogue of Corollary 4.4 (same proof) *except* path-components need not be closed (e.g. path-components of TS are T and S, but S is not closed in TS).

Corollary 4.7 *Any space is the disjoint union of its path components.* □

Relationship between the two notions

Path-connectedness $\Longrightarrow$ connectedness, but not conversely without a suitable "local" connectivity condition: A space X is locally path-connected if all points in X have "arbitrarily small" path-connected neighborhoods, i.e. $\forall x \in X$ and nbd N of x, $\exists$ path-connected nbd P of x with $P \subset N$. For example manifolds are locally path-connected (why?).

Proposition 4.8 *Every path-connected space is connected. Conversely, every locally path-connected, connected space is path-connected.*

$\boxed{\text{HW \#15}}$ Prove Proposition 4.8. (Hints: For $\Longrightarrow$ use Prop 4.3b, and for $\Longleftarrow$ show that the set of points that can be joined by a path to a given base point is a nonempty clopen set.)

Thus for manifolds the notions coincide. In general each component of a space is a union of path-components.

$\boxed{\text{HW \#16}}$ Find an example of a path-connected space which is not locally path-connected (hint: start with the topologist's sin curve).

5 Compact Spaces

Compactness is a central concept in all of math. The compact subspaces of $\mathbb{R}^n$ are exactly the closed and bounded subsets (where "bounded" means "lying in some ball") but since boundedness does not generalize to topological spaces, must use a different definition (and then this compactness criterion becomes the famous Heine-Borel Theorem).

A cover $\mathcal{U}$ of a top space X is a collection of subsets whose union is X; its an open cover if the sets are open. A subcollection of $\mathcal{U}$ which still covers X is called a subcover of $\mathcal{U}$.

Definition 5.1 X is compact if *every* open cover of X has a finite subcover.[†]

Equivalent formulation in terms of closed sets: A collection of sets has the finite intersection property (FIP) if any finite subcollection has nonempty intersection. Then X is compact iff every collection of closed subsets of X w/ the FIP has nonempty intersection (exercise).

Theorem 5.2 *Closed intervals are compact.*

Proof For any open cover $\mathcal{U}$ of an interval $[a, c]$, consider the set B of all $t \in [a, c]$ such that $[a, t]$ is covered by finitely many sets in $\mathcal{U}$. Must show $c \in B$. Set $b = \operatorname{lub} B$. Note that $b > a$ since $a \in$ some $U \in \mathcal{U} \implies [a, a+\varepsilon) \subset U$ for some $\varepsilon > 0 \implies [a, a+\varepsilon) \subset B$. Choose $V \in \mathcal{U}$ containing b and a finite subset $\mathcal{F}$ of $\mathcal{U}$ which covers $[a, t]$ for some $t < b$ in V. Then $b \in B$ (since $\mathcal{F} \cup \{U\}$ covers $[a, b]$) and $b \not< c$ (since b is an upper bound for B). Thus $b = c \in B$. $\square$

Remark Compact subsets of $\mathbb{R}^n$ can be quite wild, even for $n = 1$; e.g. the Cantor set (cf. introduction to Jänich's Topology).

Proposition 5.3 (Properties of compact spaces)

(a) *The continuous image of a compact space is compact.*

(b) *A closed subset of a compact space is compact.*

(c) *A compact subset of a Hausdorff space is closed.*

(d) *The product of two spaces is compact iff the factors are compact.*

(e) (Bolzano-Weierstrass property) *Every infinite subset of a compact space has a limit point in the space.*

Proof (a) $\mathcal{V} =$ open cover of $Y \implies f^{-1}(\mathcal{V}) = \{f^{-1}(V) : V \in \mathcal{V}\} =$ open cover of $X \implies \exists$ finite subcover $\mathcal{F} \subset f^{-1}(\mathcal{V})$ of $X \implies f(\mathcal{F}) = \{f(U) : U \in \mathcal{F}\} \subset \mathcal{V}$ is a finite subcover of $f(X)$. Thus $f(X)$ is compact. The proofs of (b) and (c) are left for HW. (d$\Longrightarrow$) is immediate from (a) since the natural projections are continuous.

[†]In other words, from any collection of open sets $\mathcal{U}$ whose union is X, one can select finitely many whose union is still equal to X. Note that a subspace A of a space X is compact $\iff$ from any collection of open sets in X whose union contains A, one can select finitely many whose union still contains A.

For (d$\Longleftarrow$) suppose we are given an open cover $\mathcal{U}$ of $S \times T$, where S and T are compact. Call a subset of $S \times T$ "good" if it lies in the union of some finite sub-collection of the sets in $\mathcal{U}$. Thus any finite union of good sets, or any subset of such a union, is good. We must show that $S \times T$ is good.

First note that every $(s,t) \in S \times T$ lies in an open box B_{st} contained in some open set in $\mathcal{U}$. In particular each B_{st} is good. For any fixed $t \in T$, the level $S \times t$ is compact (since it is homeomorphic to S which is compact) and so is covered by finitely many of these boxes. In fact this finite collection of boxes also covers the thickened level $S \times V_t$, where V_t is the (open) intersection of their projections onto T. In particular $S \times V_t$ is good. But T is also compact, and so is the union of finitely many of the V_t. Thus $S \times T$ is the union of the finitely many corresponding good sets $S \times V_t$, and so is good.

(e) Let S be an infinite subset of the compact space X. If S has no limit points in X then each $x \in X$ has a neighborhood U_x containing *at most* one point of S. But then compactness of $X \Longrightarrow X$ is covered by finitely many U_x's, contradicting that S is infinite. $\square$

$\boxed{\text{HW \#17}}$ Prove Proposition 5.3(b) and (c).

There are many consequences. For example, here is one that provides a very useful tool for constructing homeomorphisms:

Corollary 5.4 *Any bijective continuous map from a compact space to a Hausdorff space is a homeomorphism.*

Proof Let $f : X \to Y$ be such a map; ISTS f is closed (i.e. maps closed sets to closed sets). So let $A \subseteq X$ be closed. Then A is compact by Proposition 5.3(b). Thus $f(A)$ is compact by 5.3(a), and so closed in Y by 5.3(c) $\square$

We'll apply this below, after some general remarks about compactness in Euclidean space.

Compactness in $\mathbb{R}^n$

Let $A \subset \mathbb{R}^n$. If A is compact, then A is closed (by 5.3(c)) and bounded (consider the open cover consisting of all open balls about the origin). Conversely if A is closed and bounded, then it lies inside some closed "box" (i.e. product of closed intervals). But boxes are compact by 5.2 and 5.3(d), so A is compact by 5.3(b). We have proved the famous:

Heine-Borel Theorem 5.5 *A subset of $\mathbb{R}^n$ is compact $\Longleftrightarrow$ it is closed and bounded.*

$\boxed{\text{HW \#18}}$ Show how to use the Heine-Borel Theorem and Proposition 5.3(a) to deduce the Extreme Value Theorem (a key ingredient in the fundamental theorem of calculus) which asserts: *Any real valued map on a compact space achieves a maximum and minimum value.*

As another application consider the rotation group $SO(n) \subset \mathbb{R}^{n^2}$ of $\mathbb{R}^n$ (also known as the special orthogonal gp), the set of all $n \times n$ real orthogonal matrices A (i.e. $AA^t = I$) of determinant 1. (Note that an $n \times n$ matrix is naturally identified with a point in $\mathbb{R}^{n^2}$ by listing its entries in lexicographic order.) This is an example of a topological group (i.e. a group G which is also a space such that the group multiplication $G \times G \to G$ and inversion $G \to G$ are continuous).

Corollary 5.6 *$SO(n)$ is compact.*

Proof $SO(n)$ is closed (it's the intersection of two closed sets, $\det^{-1}(1)$ and $= h^{-1}(I)$, where $h(A) = AA^t$) and bounded (the columns in $A \in SO(n)$ are unit vectors). $\square$

Similarly the special unitary groups $SU(n)$ (= the set of all $n \times n$ complex unitary matrices A, meaning $AA^* = I$ where A^* denotes the conjugate transpose of A, of determinant 1) are compact.

These groups are *extremely* important in physics, especially $SO(3)$ and $SU(2)$.

Note that $SO(2)$ is homeo to S^1, via the map which sends a matrix to its first column,[†] and similarly $SU(2)$ is homeomorphic to S^3, via the map which sends a matrix to its first column (viewed as a point in $S^3 \subset \mathbb{C}^2$). Note: $SO(3)$ and $SU(2)$ are not homeomorphic, but almost; more on this later.

Remark Another way to construct interesting compact subsets of $\mathbb{R}^n$: Start with a sequence $C_1 \supset C_2 \supset \cdots$ of nonempty compact subsets of $\mathbb{R}^n$, then $C_\infty = \cap C_n$ is compact and nonempty (why? Hint: think about the Bolzano-Weierstrass property). For example, for $n = 1$ take $C_1 = I$, and inductively C_{n+1} = the complement of the open middle thirds of each of the intervals in C_n, then C_∞ is the Cantor set. Also describe the Whitehead continuum in $\mathbb{R}^3$, where each C_i is a "solid torus".

Compactness in metric spaces

By the argument above, compact $\Longrightarrow$ closed and bounded for any metric space, but the converse sometimes fails. For example the unit sphere in ℓ^2 is closed and bounded but not compact (exercise). There are, however, a number of ways to characterize compactness using metric notions.

The first pairs two important notions from analysis: A metric space X is complete if Cauchy sequences in X converge to points in X, and is totally bounded if it can be covered by a finite number of ε-balls for *any* $\varepsilon > 0$ (see §9 in Bredon or any analysis text for details). Write C/T for both conditions.

Another involves the (analytical or topological) notion of sequencial compactness, written SC, meaning that every sequence has a convergent subsequence.[†]

Theorem 5.7 *For metric spaces, the conditions* C/T, SC *and* BW *(Bolzano-Weierstrass property) are all equivalent to compactness.*

Proof See for example §9 in Bredon's *Geometry and Topology* $\square$

Remark For general top spaces, C/T doesn't make sense, and C (:= compactness) and SC are unrelated (i.e. neither implies the other). However we do have

$$\text{C} \Longrightarrow \text{BW} \Longleftarrow \text{SC}.$$

[†]this map is a homeomorphism by Corollary 5.4 since $SO(2)$ is compact, as we've just seen, and S^1 is Hausdorff.

[†]Recall that in a general topological space, a sequence x_n "converges to" x means that any neighborhood of x contains x_n for all sufficiently large n.

It is just that the reverse implications fail (see Steen and Seebach's *Counterexamples in Topology* for examples). With a suitable generalization of sequences, however, one obtains analogous necessary and sufficient conditions for compactness. (See discussion of "nets" in §6 and §7.14 in Bredon.)

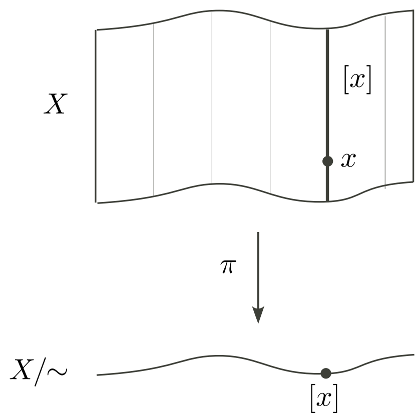
Finally we have the following useful result.

Lebesgue Covering Lemma 5.8 *For every open cover $\mathcal{U}$ of a compact metric space X, $\exists\, \lambda > 0$ such that every ball in X of radius λ lies in some $U \in \mathcal{U}$.* (Any such λ is called a Lebesgue number for the cover $\mathcal{U}$.)

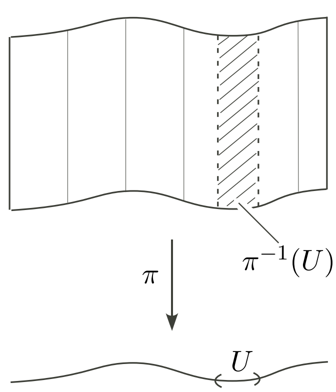
$\boxed{\text{HW \#19}}$ Prove the Lebesgue Covering Lemma. Hint: If $\nexists\, \lambda$, then for each integer $n > 0$, there is a ball B_n of radius $1/n$ not lying in any $U \in \mathcal{U}$. Show how this leads to a contradiction by considering the set of all centers of these balls and its limit points.

6 Quotient Spaces

Let X be a set with an equivalence relation $\sim$. Write $X/\sim$ for the set of all equivalence classes, and $\pi : X \to X/\sim$ for the natural projection sending each $x \in X$ to its equivalence class $[x]$. Note that $[x]$ can either be viewed as a point in $X/\sim$ or as a subset of X (the latter is the preimage of the former under π). Here is a picture in which each equivalence class is represented by a vertical line segment which π maps to the point directly below it:



Now if X is a topological space, then $X/\sim$ has a natural topology, called the quotient topology, consisting of all the subsets $U \subset X/\sim$ for which $\pi^{-1}(U)$ is open in X. Note that $\pi^{-1}(U)$ is a "saturated" open set in X in the sense that it is a union of equivalence classes, so the *open sets in* $X/\sim$ naturally correspond to the *saturated open sets in* X. Evidently the projection π is continuous when $X/\sim$ is given this topology; in fact this is the largest topology on $X/\sim$ for which π is continuous. Endowed with this topology, $X/\sim$ is called a quotient space of X.



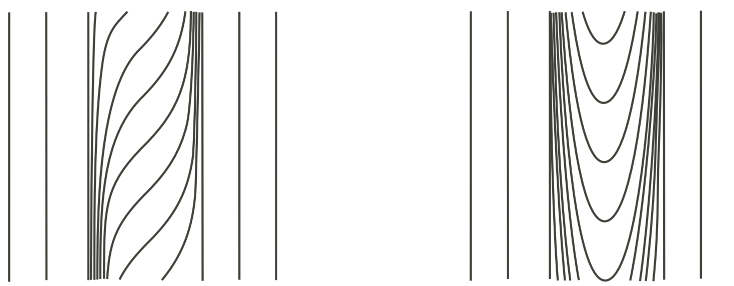
Reiterating: the open sets in $X/\sim$ are exactly the subsets whose preimages under π are open in X. And it follows that the closed sets in $X/\sim$ are exactly the subsets whose preimages under π are closed in X; in particular a point $[x] \in X/\sim$ is closed (when viewed as a one-element subset of $X/\sim$) if and only the equivalence class $[x]$ is a closed subset of X.

Properties inherited by quotient spaces

If the space X is connected, path-connected, or compact, then $X/\sim$ will have the same property. This follows from Propositions 3.3a, 3.6a and 4.3a since $X/\sim$ is the image of X under the continuous map π.

However this is not the case for the Hausdorff property: X being Hausdorff does not force $X/\sim$ to be Hausdorff. For if $X/\sim$ is Hausdorff then it is T_1 and so all its points are closed, or equivalently all the equivalence classes of $\sim$ must be closed subsets of X. For example the two-element space $[0,1]/\sim$, where the equivalence classes of $\sim$ are $\{0\}$ and $(0,1]$, is not Hausdorff since $(0,1]$ is not a closed subset of $[0,1]$. Conversely, all equivalence classes being closed does *not* generally force $X/\sim$ to be Hausdorff, as seen by the following example.

$\boxed{\text{HW \#20}}$ Consider the two equivalence relations $\sim_1$, $\sim_2$ on $\mathbb{R}^2$ defined as follows: Both have the vertical lines $x = c$ as equivalence classes for all c with $|c| \geq \pi/2$. The other equivalence classes for $\sim_1$ (resp. $\sim_2$) are the graphs of $\tan(x) + c$ (resp. $\sec(x) + c$) on $(-\pi/2, \pi/2)$, for each $c \in \mathbb{R}$. Note that all these equivalence classes are closed subsets of $\mathbb{R}^2$.



Show that exactly one of the spaces $X_i = \mathbb{R}^2/\sim_i$ (for $i = 1, 2$) is Hausdorff. (Hint: In thinking about separating a pair of equivalence classes, recall that the open sets in X_i can be thought of as saturated open sets in $\mathbb{R}^2$.)

Universal property of quotient spaces

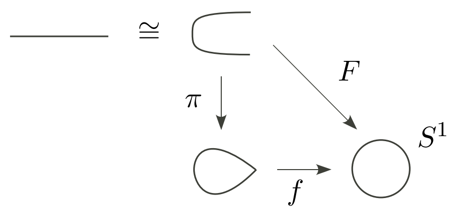
Theorem 6.1 Let $X/\sim$ be a quotient space. Then for any map $F : X \to Y$ which is constant on equivalence classes of $\sim$, there exists a unique map $f : X/\sim \to Y$ such that $F = f \circ \pi$, where $\pi : X \to X/\sim$ is the canonical projection, i.e. such that the following diagram commutes:

$$\begin{tikzcd} X \arrow[d, "\pi"'] \arrow[dr, "F"] & \\ X/\sim \arrow[r, dashed, "f"'] & Y \end{tikzcd}$$

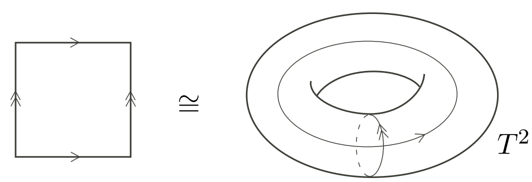
Proof It is clear that there is a unique such function f, namely the one which maps each equivalence class to the image under F of any point in that class (i.e. $f([x]) = F(x)$). We must only show that f is continuous. But U open in $Y \Longrightarrow F^{-1}(U) = (f \circ \pi)^{-1}(U) = \pi^{-1}(f^{-1}(U))$ which is open in X. Thus $f^{-1}(U)$ is open in $X/\sim$ by definition of the quotient topology. $\square$

This theorem is most often used to construct familiar models for quotient spaces.

Example 6.2 The space obtained by identifying the endpoints of the closed interval $I = [0,1]$ (meaning $I/0 \sim 1$, i.e. the equivalence classes are $\{0,1\}$ and all the singletons $\{t\}$ for $t \in (0,1)$) is homeomorphic to the circle S^1. Indeed the map $F : I \to S^1$, $t \mapsto (\cos 2\pi t, \sin 2\pi t)$ induces a map $f : I/{\sim} \to S^1$ (by Theorem 6.1) which is clearly a bijection, and therefore a homeomorphism by Corollary 5.4. (Note that $I/{\sim}$ is compact since I is, and S^1 is Hausdorff.)



Similarly the space obtained by identifying opposite sides of a square $I \times I$ as indicated below (meaning $I \times I/(t,0) \sim (t,1)$, $(0,t) \sim (1,t)$ for all $t \in I$) is homeomorphic to the torus T^2.



Examples of quotient spaces

Two important types of quotient spaces are homogeneous spaces G/H (whose points are the cosets of a subgroup H of a topological group G) and orbit spaces X/G (whose points are the orbits of a continuous action of a topological group G on a topological space X; see Janich for definitions). For example

1. The meridian in T^2 (marked with a double arrow above) is a subgroup isomorphic[†] to the circle S^1 (namely $1 \times S^1$ if T^2 is identified with $S^1 \times S^1$) with quotient $T^2/S^1 \cong S^1$ (another group in this case, although in general homogeneous spaces are not groups).

2. The action of S^1 on S^2 by rotations about the z-axis has orbit space $S^2/S^1 \cong I$.

(Draw pictures.) These facts can be proved rigorously using the universal property as in Example 5.2, applied to the projections of $T^2 = S^1 \times S^1$ onto its first factor in the former case, and of S^2 onto the z-axis in the latter.

[†]as a topological group, i.e. isomorphic as a group and homeomorphic as a space

One particularly important example of an orbit space is the projective space $\mathbb{R}P^n$ (for any integer $n \geq 0$) defined as the quotient of the space of nonzero vectors in $\mathbb{R}^{n+1}$ by the multiplication action of the group of nonzero real numbers. The orbits are the (punctured) lines through 0 in $\mathbb{R}^{n+1}$, and so

$$\mathbb{R}P^n = \frac{\mathbb{R}^{n+1} - 0}{x \sim \lambda x} \quad \text{(for all } \lambda \neq 0 \text{ in } \mathbb{R}).$$

It is easy to check by hand that $\mathbb{R}P^n$ is Hausdorff. It is also compact, which can be checked from the alternative descriptions

$$\mathbb{R}P^n \cong \frac{S^n}{x \sim -x} \cong \frac{B^n}{x \sim -x \ \text{ for } \ x \in \partial B^n}.$$

which follow from the following basic principle:

Lemma 6.3 *If the quotient space $X/\sim$ is Hausdorff and A is a compact subset of X that contains at least one point in each equivalence class of $\sim$, then $X/\sim \cong A/\sim$.*

$\boxed{\text{HW \#21}}$ Prove Lemma 6.3 using the universal property of quotient spaces.

See pages 34–39 in Janich for some more sophisticated remarks about homogeneous spaces and orbit spaces. We shall not discuss them any more for now.

Two other common ways to construct quotients are (1) collapsing subspaces of a space, and (2) gluing spaces together (see pages 39–49 in Janich):

Collapsing

Let A be a subspace of a space X. Then X/A is the quotient space obtained by collapsing A to a single point a, i.e. $X/A := X/\sim$ where the equivalence classes of $\sim$ are A and all the singletons $\{x\}$ for $x \in X - A$.

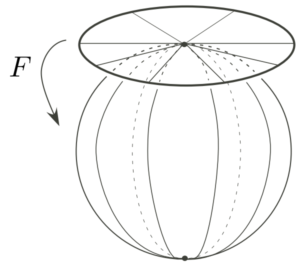
More generally if $A = A_1 \sqcup \cdots \sqcup A_k$, a disjoint union of a finite number $k > 1$ of subspaces of X, then define $X/A_1, \ldots, A_n$ to be the space obtained from X by collapsing the sets $A_1, \ldots, A_n$ separately to k distinct points $a_1, \ldots, a_k$.

Examples

① We have already seen the example $I/\{0,1\} \cong S^1$ (in 6.2 above), or equivalently $B^1/\partial B^1 \cong S^1$ (since there is a homeomorphism $I \to B^1 = [-1,1]$ carrying $\{0,1\}$ onto $\partial B^1 = S^0$). Similarly

$$B^2/\partial B^2 \cong S^2.$$

Indeed there is a natural map $F : B^2 \to S^2$ which sends the origin to the north pole, and wraps the rest of B^2 around S^2, sending ∂B^2 to the south pole. For example, F can be given by the formula $F(r,\theta) = (\theta, 2\pi r)$ (using polar coordinates (r,θ) on B^2 and spherical coordinates (θ,ϕ) on S^2). In words, F sends each radial segment onto a corresponding longitude (i.e. half great circle) on S^2 from the north pole to the south pole. Here's a picture:



This map induces a continuous bijection (and thus a homeomorphism) $B^2/\partial B^2 \to S^2$. More generally

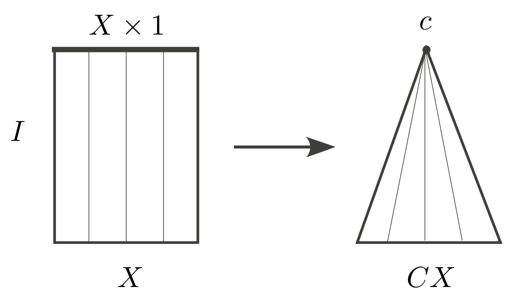
$$B^n/\partial B^n \cong S^n$$

by the same proof (where we happily accept the verbal description of the analogous map F, knowing however that we could write down a formula for F if put to the test).

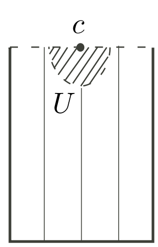
② The <u>cone</u> on a space X is the quotient

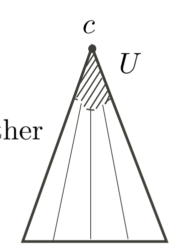
$$CX := X \times I \,/\, X \times 1.$$

In other words all the points on the "top" $X \times 1$ of $X \times I$ are collapsed to a single point c (called the "cone point") as shown below.



Note that CX should not be visualized as on the left below, since this gives the wrong image of the open neighborhoods U of the cone point. The picture on the right shows the correct image.

not 

, but rather 

For example

$$CB^n \cong B^{n+1} \qquad \text{and} \qquad CS^n \cong B^{n+1}$$

shown using the universal property applied to the map

$$B^n \times I \to B^{n+1},\ (x,t) \mapsto t(0,1) + (1-t)(x, -\sqrt{1-\|x\|^2})$$

in the first case (viewing $B^{n+1} \subset \mathbb{R}^{n+1} = \mathbb{R}^n \times \mathbb{R}$), and

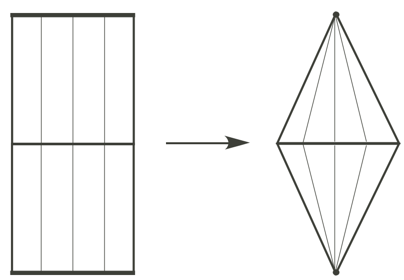
$$S^n \times I \to B^{n+1},\ (x,t) \mapsto (1-t)x$$

in the second.

③ The suspension of a space X is the quotient

$$\Sigma X := X \times I \,/\, X \times 0,\ X \times 1$$

In other words all the points on the "bottom" $X \times 0$ of $X \times I$ are collapsed to a point s_0, and all the points on the top $X \times 1$ are collapsed to another point s_1; these two points are called the "suspension points".



For example $\Sigma B^n \cong B^{n+1}$ (exercise) and:

$\boxed{\text{HW \#22}}$ Show that $\Sigma S^n \cong S^{n+1}$.

Gluing

MORE TO COME ... At some point, consider the case when Hausdorff spaces X_1 and X_2 are glued together via a homeomorphism $h : U_1 \to U_2$ where the U_i are open in X_i (this comes up when gluing smooth manifolds together to get a smooth manifold, e.g. in the definition of connected sum). In that case assign a HW that asks for necessary and sufficient conditions for the result to be Hausdorff.[†]

[†]A sufficient condition is that the $X_i - U_i$ have open neighborhoods V_i in X_i such that $h(V_1) \cap V_2 = \varnothing$. Probably not necessary. Necessary and sufficient condition is that for all points $x_i \in \partial U_i$, there should be neigborhoods V_i of x_i in X_i such that $h(V_1) \cap V_2 = \varnothing$.