COMPLEX ANALYSIS

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1. Analytic Functions

A. Complex Numbers

Definition A complex number is an ordered pair (x, y) of real numbers, that is, a vector in $\mathbb{R}^2$. One adds complex numbers using vector addition

$$(x_1, y_1) + (x_2, y_2) \;=\; (x_1 + x_2\,,\; y_1 + y_2)$$

and multiplies them using the interesting formula

$$(x_1, y_1)(x_2, y_2) \;=\; (x_1 x_2 - y_1 y_2\,,\; x_1 y_2 + x_2 y_1)$$

which amounts to the rule "multiply lengths and add angles".[†] (More on this below)

Exercise Addition and multiplication are *associative* and *commutative* operations with *identities* $(0, 0)$ and $(1, 0)$, resp., and multiplication *distributes* over addition. Furthermore, any complex number (x, y) has an *additive inverse* (or *negative*) $(-x, -y)$.

Denote the set of all complex numbers by $\mathbb{C}$, and view $\mathbb{R} \subset \mathbb{C}$ by identifiying $x \in \mathbb{R}$ with $(x, 0) \in \mathbb{C}$. Since $(x_1, 0) + (x_2, 0) = (x_1 + x_2, 0)$ and $(x_1, 0)(x_2, 0) = (x_1 x_2, 0)$, addition and multiplication in $\mathbb{R}$ are the same whether performed before or after this identification is made. Also $(0, 0) = 0$ and $(1, 0) = 1$, as the identities for $+$ and $\cdot$ are usually written.

Setting $i = (0, 1)$, it is straightforward to check that

$$(x, y) \;=\; x + iy$$

which is the traditional way to write complex numbers. Also $i^2 = -1$, and sums and products can be computed in the familiar way $(x_1 + iy_1) + (x_2 + iy_2) = (x_1 + x_2) + i(y_1 + y_2)$ and $(x_1 + iy_1)(x_2 + iy_2) = x_1 x_2 + x_1 iy_2 + iy_1 x_2 + iy_1 iy_2 = (x_1 x_2 - y_1 y_2) + i(x_1 y_2 + x_2 y_1)$.

Definition Let $z = x + iy$ be a complex number. We call x and y the real and imaginary parts of z, written $\mathrm{Re}(z)$ and $\mathrm{Im}(z)$. If (r, θ) are the polar coordinates of the point (x, y), then we call r and θ the norm (or modulus) and argument of z, written $|z|$ and $\arg(z)$. Thus

$$|z| \;=\; r \;=\; \sqrt{x^2 + y^2} \qquad \text{and} \qquad \arg(z) \;=\; \theta \;=\; \arctan(y/x)\,.$$

Note that $r \geqslant 0$, and θ is defined and multivalued when $r > 0$. For example $|1 + i| = \sqrt{2}$ and $\arg(1 + i)$ is equal to $\pi/4 + 2\pi n$ for any $n \in \mathbb{Z}$.

[†] This means that in polar coordinates $(r_1, \theta_1)(r_2, \theta_2) = (r_1 r_2, \theta_1 + \theta_2)$. To see this, compute the product $(x_1, y_1)(x_2, y_2) = (r_1 \cos\theta_1, r_1 \sin\theta_1)(r_2 \cos\theta_2, r_2 \sin\theta_2)$. By definition this is given by the formula

$$(r_1 r_2(\cos\theta_1 \cos\theta_2 - \sin\theta_1 \sin\theta_2)\,,\; r_1 r_2(\cos\theta_1 \sin\theta_2 + \sin\theta_1 \cos\theta_2))$$

which is equal to $(r_1 r_2 \cos(\theta_1 + \theta_2)\,,\; r_1 r_2 \sin(\theta_1 + \theta_2))$ by trigonometry.

The following basic properties are readily verified for any pair z, w of complex numbers

$$\begin{aligned} \mathrm{Re}(z+w) &= \mathrm{Re}(z)+\mathrm{Re}(w) & |zw| &= |z||w| \\ \mathrm{Im}(z+w) &= \mathrm{Im}(z)+\mathrm{Im}(w) & \arg(zw) &= \arg(z)+\arg(w) \pmod{2\pi} \end{aligned}$$

the ones on the right being the "multiply lengths and add angles" rule for multiplication. We also have the important triangle inequality

$$|z+w| \leqslant |z|+|w|$$

which is geometrically obvious, or see Proposition 1.2.5 in MH (Marsden-Hoffman's text) for an algebraic proof, and its useful consequence: $|z \pm w| \geqslant ||z|-|w||$ (exercise).

Define the conjugate of $z = x+iy$ to be $\overline{z} = x-iy$ (geometrically $\overline{z}$ is the reflection of z through the x-axis) It is easy to verify the formulas (cf. Proposition 1.2.4 in MH)

$$\overline{z+w} = \overline{z}+\overline{w}\,, \quad \overline{zw} = \overline{z}\,\overline{w} \qquad \text{and} \qquad \overline{\overline{z}} = z$$

which show that conjugation is an "involution" of $\mathbb{C}$,† and

$$\mathrm{Re}(z) = (z+\overline{z})/2\,, \quad \mathrm{Im}(z) = (z-\overline{z})/2i \qquad \text{and} \qquad z\overline{z} = |z|^2\,.$$

The last formula implies that any nonzero complex number z has a multiplicative inverse

$$z^{-1} = \overline{z}/|z|^2.$$

With the exercise above, this proves:

1.1 Theorem $\mathbb{C}$ *is a field.* (Theorem 1.1.2 in MH)

Polar Form Given any complex number $z = x+iy$, we have $x = r\cos\theta$ and $y = r\sin\theta$, where $r = |z|$ and $\theta = \arg(z)$, so $z = r(\cos\theta + i\sin\theta)$. Using Euler's Identity

$$e^{i\theta} = \cos\theta + i\sin\theta$$

(motivated in the next section) we can rewrite z in polar form

$$z = re^{i\theta}$$

which is very convenient for many purposes. For example products in polar form become

$$(*) \qquad re^{i\theta}\, se^{i\varphi} = rs\, e^{i(\theta+\varphi)}$$

by the "multiply lengths and add angles" rule, or as one would expect using the familiar laws of exponents. This is illustrated below

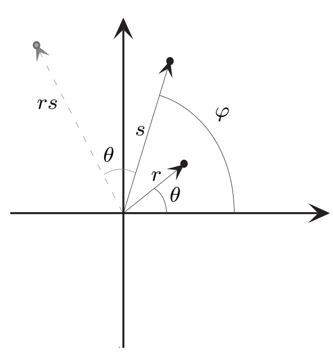

FIGURE 1. Complex multiplication

† An involution of $\mathbb{C}$ is by definition a homomorphism $f : \mathbb{C} \to \mathbb{C}$ (meaning $f(z+w) = f(z)+f(w)$ and $f(zw) = f(z)f(w)$) that satisfies $f \circ f = \mathrm{id}$. These properties are clearly satisfied by $f(z) = \overline{z}$.

Applications $(*)$ implies the formulas for the sine and cosine of the sum of two angles; just take $r = s = 1$, and expand both sides using Euler's identity. It also yields identities for sines and cosines of multiples of angles. For example, $e^{i(3\theta)} = \cos 3\theta + i \sin 3\theta = (e^{i\theta})^3 = (\cos\theta + i\sin\theta)^3 = \cos^3\theta + 3i\cos\theta\sin\theta - 3\cos\theta\sin^2\theta - i\sin^3\theta$, so

$$\cos 3\theta \;=\; \cos^3\theta - 3\cos\theta\sin^2\theta \qquad \text{and} \qquad \sin 3\theta \;=\; 3\cos\theta\sin\theta - \sin^3\theta\,.$$

As another application, consider the problem of computing the powers or the roots of a complex number $z = re^{i\theta}$. Applying $(*)$ repeatedly when $re^{i\theta} = se^{i\phi}$, and using Euler's identity, we obtain <u>DeMoivre's</u> <u>Formula</u> for the <u>powers</u> of z:

$$z^n \;=\; r^n e^{in\theta} \;=\; r^n(\cos n\theta + i\sin n\theta)$$

for any integer $n \geqslant 0$, and consequently for $n < 0$ as well, since $z^{-1} = r^{-1}e^{-i\theta}$. In fact this holds for any rational number in place of n, when properly interpreted. For example, to compute the nth roots $se^{i\varphi}$ of $z = re^{i\theta}$, we have $s^n e^{in\varphi} = re^{i\theta}$, and so $s^n = r$ and $n\varphi = \theta \pmod{2\pi}$. Thus z <u>has</u> <u>exactly</u> n <u>distinct</u> nth <u>roots</u>:

$$z^{1/n} \;=\; r^{1/n}e^{i(\theta+2\pi k)/n} \;=\; r^{1/n}\left(\cos\left((\theta+2\pi k)/n\right) + i\sin\left((\theta+2\pi k)/n\right)\right)$$

for $k = 0, 1, \ldots, n-1$. Alternatively, these roots can be written as $\nu, \nu\omega, \ldots, \nu\omega^{n-1}$ where $\nu = r^{1/n}e^{i\theta/n}$ and $\omega = e^{2\pi i/n}$. They are equally distributed on a circle of radius $r^{1/n}$ about 0, since multiplication by ω rotates $\mathbb{C}$ about 0 by $2\pi/n$ radians (verify this). For example, the cubes and cube roots of $(1+i) = 2^{1/2}e^{i\pi/4}$ are

$$(1+i)^3 \;=\; 2^{3/2}e^{3\pi i/4} \qquad \text{and} \qquad (1+i)^{1/3} \;=\; 2^{1/6}e^{\pi i/12},\; 2^{1/6}e^{9\pi i/12} \;\text{ or }\; 2^{1/6}e^{17\pi i/12}.$$

<u>B. Complex Functions</u>

<u>Definition</u> A <u>complex</u> <u>function</u> is a function $f : A \to \mathbb{C}$ with <u>domain</u> $A \subset \mathbb{C}$. Thus

$$f(x+iy) \;=\; u(x,y) + iv(x,y)$$

for suitable real valued functions u and v on $A \subset \mathbb{R}^2$ (identifying $\mathbb{C}$ with $\mathbb{R}^2$). We call u and v the <u>real</u> and <u>imaginary</u> <u>parts</u> of f, and often simply write $f = u + iv$.

<u>Examples</u> $\boxed{\text{Complex multiplication}}$ For fixed $z_0 = x_0 + iy_0$, define $m_{z_0} : \mathbb{C} \to \mathbb{C}$ by

$$m_{z_0}(z) \;=\; z_0 z \;=\; (x_0x - y_0y) + i(y_0x + x_0y) \qquad (\text{for } z = x + iy)$$

<u>Geometry</u>: Dilate by $|z_0|$ and rotate by $\arg z_0$ (about the origin)

<u>Linear Algebra</u>: Linearly transforms $\mathbb{R}^2$ by multiplying by $\begin{pmatrix} x_0 & -y_0 \\ y_0 & x_0 \end{pmatrix}$

$\boxed{\text{The exponential function}}$ Define $\exp : \mathbb{C} \to \mathbb{C}$ by

$$\exp(z) \;=\; e^z \;:=\; e^x(\cos y + i\sin y) \qquad (\text{for } z = x + iy)$$

Thus $|e^z| = e^x$ and $\arg(e^z) = y$. This extends the usual exponential function when the variable is real, and yields Euler's identity when it is purely imaginary.

<u>Motivation</u>: Recall $e^x = 1 + x + x^2/2! + x^3/3! + \cdots$ for $x \in \mathbb{R}$. Thus it is natural to let

$$e^{iy} \;=\; 1 + iy + \frac{(iy)^2}{2!} + \frac{(iy)^3}{3!} + \frac{(iy)^4}{4!} + \cdots \;=\; (1 - \frac{y^2}{2!} + \frac{y^4}{4!}\cdots) + i\,(y - \frac{y^3}{3!} + \cdots)$$

which equals $\cos y + i\sin y$. In general we want $e^{x+iy} = e^x e^{iy}$, which yields the definition above.

Properties: (a) (law of exponents) $e^{z+w} = e^z e^w$

(b) (periodicity) exp has period $2\pi i$, i.e. $e^{z+p} = e^z \iff p$ is an integral multiple of $2\pi i$

Proofs: (a) We show both sides have the same norm and argument. Set $z = x + iy$ and $w = u + iv$, so $z + w = (x + u) + i(y + v)$. Then $|e^{z+w}| = e^{x+u} = e^x e^u = |e^z||e^w| = |e^z e^w|$ and $\arg(e^{z+w}) = y + v = \arg(e^z) + \arg(e^w) = \arg(e^z e^w)$. (b) $e^{z+p} = e^z \iff e^p = 1 \iff \mathrm{Re}(p) = 0$ and $\mathrm{Im}(p) \in 2\pi\mathbb{Z}$ (by the known periodicity of the real functions sin and cos) $\iff p \in 2\pi i\mathbb{Z}$. Geometric proof below.

Geometry: The x-axis maps to the positive x-axis by the usual exponential map. In fact all the horizontal lines $y = c$ map to *open* rays $\theta = c$ emanating from 0, sweeping around with period 2π in θ (so in particular, exp has image $\mathbb{C} - \{0\}$). The vertical lines $x = c$ map to circles of radius e^c centered at 0. Here is a "dynamic" picture of exp:

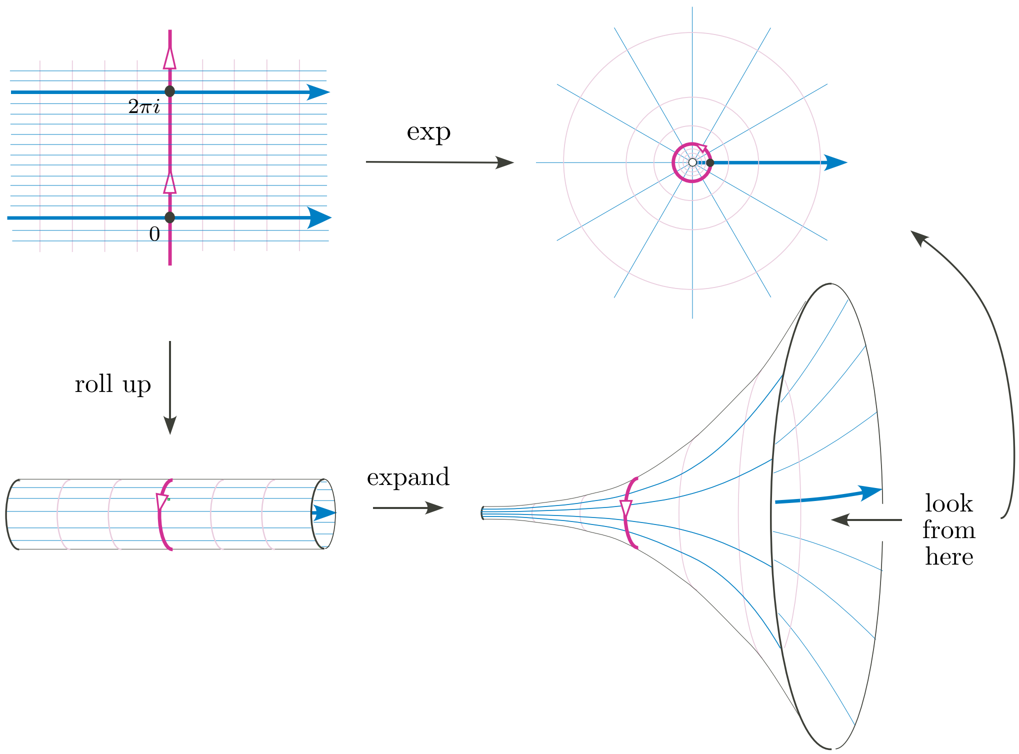

Figure 2. The exponential function

The logarithm Define $\log : \mathbb{C} - \{0\} \longrightarrow \mathbb{C}$ by

$$\log z \;=\; \log|z| + i \arg z.$$

As it stands, this is a multivalued function, since arg is multivalued. If we restrict the domain to the complement of the negative real axis, however, we obtain a continuous (singlevalued) function

$$\log : \{z \,:\, \arg z \in (-\pi, \pi)\} \longrightarrow \{z \,:\, \mathrm{Im}(z) \in (-\pi, \pi)\}$$

that extends the usual log function on $\mathbb{R}$. This is called the principal branch of the logarithm. Other branches are obtained by restricting to the complement of a different ray emanating from the origin (i.e. restrict $\arg z$ to another open interval of length 2π). Thus any $z \neq 0$ is in the domain of some branch of the logarithm.

Motivation: Want log to be the inverse function of exp, i.e. $e^{\log z} = z$. Setting $\log z = x+iy$, this makes $z = e^x e^{iy} \implies e^x = |z|$, i.e. $x = \log|z|$, and $y = \arg z$, as defined above.

Property: $\log(zw) = \log z + \log w$.

Proof: $\log(zw) = \log|zw| + i\arg(zw) = \log(|z||w|) + i(\arg(z) + \arg(w)) = \log|z| + \log|w| + i\arg z + i\arg w = \log z + \log w$.

Geometry:

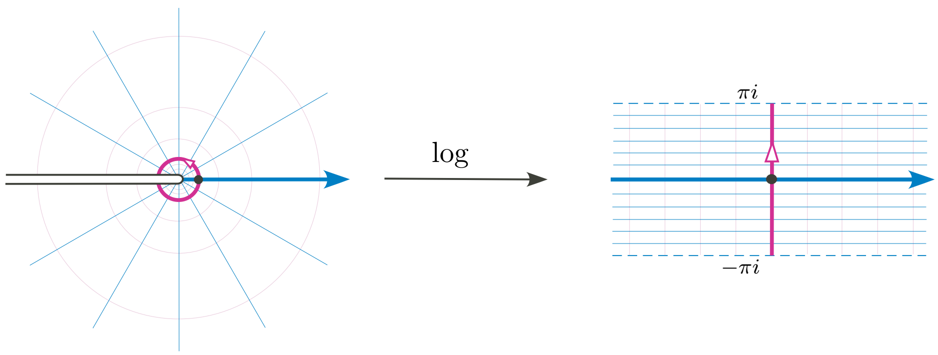

FIGURE 3. The logarithm function (principal branch)

$\boxed{\text{Trigonometric functions}}$ Define $\sin, \cos, \sinh, \cosh : \mathbb{C} \to \mathbb{C}$ by

$$\sin z = \frac{e^{iz} - e^{-iz}}{2i}, \quad \cos z = \frac{e^{iz} + e^{-iz}}{2}, \quad \sinh z = \frac{e^z - e^{-z}}{2}, \quad \cosh z = \frac{e^z + e^{-z}}{2}$$

Motivation: If $x \in \mathbb{R}$ then $e^{ix} - e^{-ix} = (\cos x + i\sin x) - (\cos(-x) + i\sin(-x)) = 2i\sin x$ so $\sin x = (e^{ix} - e^{-ix})/2i$, and similarly $\cos x = (e^{ix} + e^{-ix})/2$. For sinh and cosh, these are just the usual definitions from calculus. The graphs of these real functions are:

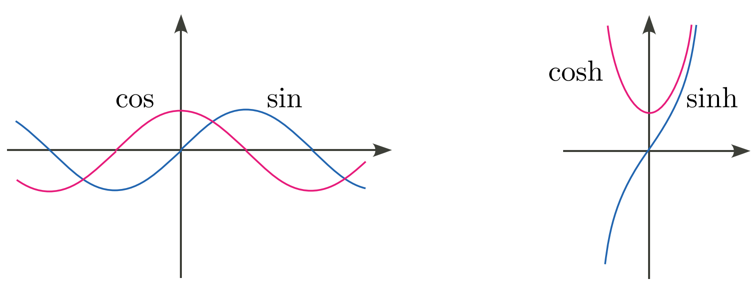

FIGURE 4. The real trigonometric functions

Properties: (a) $\sin(z+w) = \sin z\cos w + \cos z \sin w$, $\cos(z+w) = \cos z\cos w - \sin z\sin w$, $\sinh(z+w) = \sinh z\cosh w + \cosh z\sinh w$, $\cosh(z+w) = \cosh z\cosh w + \sinh z\sinh w$.

(b) $\cos^2 x + \sin^2 x = 1$, $\cosh^2 x - \sinh^2 x = 1$

(c) $\cos z = \sin(z + \pi/2)$, $\sinh z = -i\sin(iz)$, $\cosh(z) = \cos(iz)$

(d) sin and cos are periodic of period 2π, and sinh and cosh are periodic of period $2\pi i$

Proofs: Properties (a), (b) and (c) are straightforward from the definitions (for (a) start with the right-hand side). Property (d) also follows from the definitions and the fact that exp has period $2\pi i$. The details are left to the reader.

Geometry: We discuss sin, leaving cos for homework, and sinh and cosh as exercises. (Note: property (c) above shows that the pictures are closely related.) Using (a), write

$$\begin{aligned}\sin(x+iy) &= \sin(x)\cos(iy)+\cos(x)\sin(iy)\\ &= (\sin x\cosh y)+i(\cos x\sinh y) \;=\; u+iv.\end{aligned}$$

For each $a, b \in \mathbb{R}$, let V_a be the upward-pointing vertical line $\{x+iy : x=a\}$, and H_b be the right-pointing horizontal line $\{x+iy : y=b\}$. Where does the sine function map V_a and H_b in the uv-plane?

The real axis H_0 clearly maps to the interval $[-1,1]$ in the u-axis by the real function sin. For $b \neq 0$ we have $\sin(x+ib) = (\sin x\cosh b)+i(\cos x\sinh b)$, so H_b maps to the ellipse

$$\frac{u^2}{\cosh^2 b}+\frac{v^2}{\sinh^2 b} \;=\; 1$$

since $\sin^2+\cos^2 = 1$. These ellipses enclose $[-1,1]$ since $\cosh \geqslant 1$. They are traversed clockwise when $b>0$ and counterclockwise when $b<0$ (check what happens near $x=0$) and grow in size as $|b|$ grows.

As for the vertical lines, consider V_a for $a = n\pi/2$ for $n \in \mathbb{Z}$. If n is even then $\sin(a+iy) = \pm i\sinh y$, so V_a maps to the v-axis, pointing up or down according to whether $n \equiv 0$ or $2 \bmod 4$. If n is odd then $\sin(a+iy) = \pm\cosh y$, and so V_a maps (folded in half) to the ray $u \geqslant 1$, or $u \leqslant -1$ on the u-axis, according to whether $n \equiv 1$ or $3 \bmod 4$. For all other values of a, the line V_a maps to one branch of the hyperbola

$$\frac{u^2}{\sin^2 a}-\frac{v^2}{\cos^2 a} \;=\; 1$$

since $\cosh^2 - \sinh^2 = 1$.

The picture is:

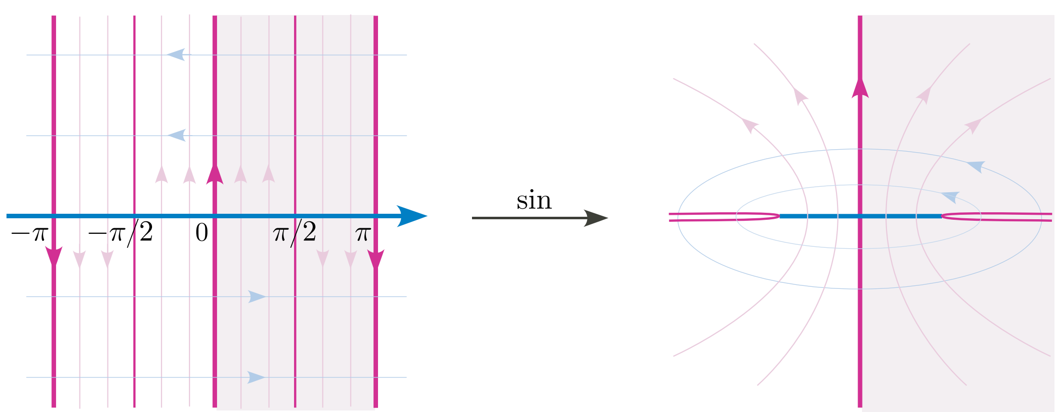

Figure 5. The sine function

Problem Find the maximum of $|\sin z|$ on the square $\{z : \mathrm{Re}(z) \text{ and } \mathrm{Im}(z) \in [0,\pi]\}$.

Solution: The sine function maps the horizontal segments $x+ib$, for $x \in [0,\pi]$, to the right halves of the ellipses $u^2/\cosh^2 b + v^2/\sinh^2 b = 1$ in the uv-plane, starting at

$(0, \sinh b)$ and ending at $(0, -\sinh b)$. These ellipses are nested, increasing in size with b. Since $\cosh^2 b > \sinh^2 b$, it follows that $|\sin|$ achieves a maximum value of $\cosh \pi$ at the midpoint $\pi/2 + i\pi$ of the upper edge of the square.

$\boxed{\text{Complex powers}}$ For any two complex numbers $b \neq 0$ and p, define the complex number

$$b^p \;=\; e^{p \log b}.$$

This is in general multivalued, since log is multivalued, with distinct values of b^p differing by powers of $e^{2\pi i p}$. In particular, if $p = n/d \in \mathbb{Q}$ (in lowest terms with $n \geqslant 0$) then b^p takes on exactly d values, namely the dth roots of $b^n = b \cdots b$ (n times). In all other cases b^p takes on infinitely many values. So b^p is single valued iff $p \in \mathbb{Z}$.

Exercises Show (a) $\log(b^p) = p \log b$, and (b) $(b^p)^q = b^{pq}$.

There are two kinds of associated functions: power functions (taking b as the variable) and exponential functions (taking p as the variable):

For fixed $p \in \mathbb{C}$ define $\mathrm{pow}_p : \mathbb{C} - \{0\} \to \mathbb{C}$ by

$$\mathrm{pow}_p(z) \;=\; z^p \;=\; e^{p \log z}.$$

As noted above, this is multivalued unless $p \in \mathbb{Z}$. The most important special cases are $p = n$ (nth power) and $p = 1/n$ (nth roots) for $n \in \mathbb{Z}$.

For fixed nonzero $b \in \mathbb{C}$ define $\exp_b : \mathbb{C} \to \mathbb{C}$ by

$$\exp_b(z) \;=\; b^z \;=\; e^{z \log b}.$$

This is always multivalued, taking on d values when z is rational of the form n/d (in lowest terms) and infinitely many values for all other z.

Exercises Give geometric descriptions of (a) pow_n and $\mathrm{pow}_{1/n}$ for $n \in \mathbb{Z}$, and (b) $\exp_b$ (hint: use the descriptions for m_b and exp above)

C. Continuity

We begin with some basic "topology" in the plane. For any positive real number r and point $a \in \mathbb{C}$, define the open disk

$$D_r(a) \;=\; \{z \in \mathbb{C} \;:\; |z - a| < r\} \qquad \text{(denoted } D(a; r) \text{ in MH)}$$

consisting of all points in $\mathbb{C}$ at distance strictly less than r (the radius) from a (the center). Also consider the associated closed disk and punctured open disk, defined by

$$\overline{D}_r(a) \;=\; \{z \in \mathbb{C} \;:\; |z - a| \leqslant r\} \qquad \text{and} \qquad \check{D}_r(a) \;=\; \{z \in \mathbb{C} \;:\; |0 < |z - a| < r\}$$

respectively. These disks are sketched below (exclude the points on the dotted boundaries).

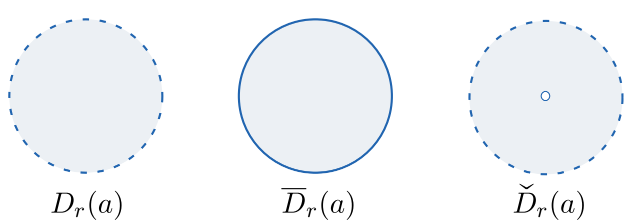

Figure 6. Disks

If $r = 1$ or $a = 0$, we may sometimes omit them from the notation. For example D, unless otherwise specified, will denote the unit disk, of radius 1 centered at the origin 0. These disks will help us understand the notions of open and closed subsets of $\mathbb{C}$, limits of complex functions and sequences, and continuous and differentiable functions.

Open and closed sets

Definition A subset A of $\mathbb{C}$ is open if each point in A is the center of some open disk lying entirely inside A, or in symbols

$$\forall\, a \in A, \exists\, r > 0 \text{ such that } D_r(a) \subset A,$$

and is closed if its complement $\mathbb{C} - A$ is open, i.e. $\forall\, a \notin A, \exists\, r > 0$ such that $D_r(a) \subset \mathbb{C} - A$.

Exercise Show (using the triangle inequality) that open disks are open, and closed disks are closed. Also show that open disks are not closed, and closed disks are not open.

More generally, the region *strictly inside* – or *strictly outside* – a smooth simple closed curve in $\mathbb{C}$ is open. If one includes the points on the curve, then the resulting sets are closed, since their complements will be open. Examples of such regions are sketched below.

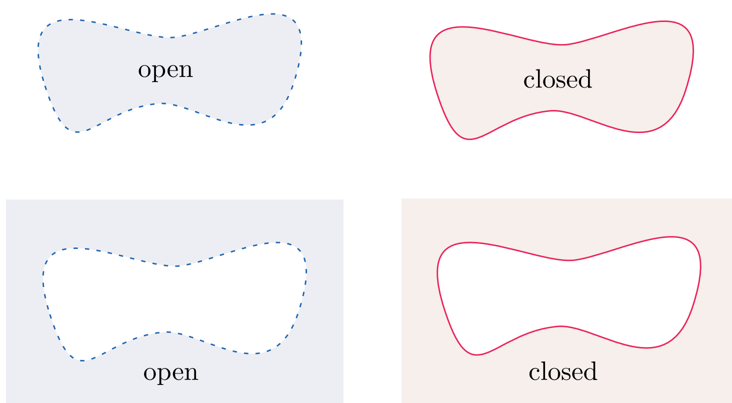

Figure 7. Open and closed sets

Needless to say, there are many other kinds of open and closed sets in $\mathbb{C}$, but we will not discuss these now.

Exercises (a) Show that the union or intersection of any two open sets is open, and that the union or intersection of any two closed sets is closed.[†]

(b) Show that the only subsets of $\mathbb{C}$ that are both open and closed (clopen for short) are $\mathbb{C}$ itself and the empty set $\varnothing$.

(c) Show that a subset of $\mathbb{C}$ is open if and only if it is a union of (possibly infinitely many) open disks, and closed if and only if it contains all its limit points. Here a point a is called a limit point of $A \subset \mathbb{C}$ if there are points in A "arbitrarily close" to a, i.e. if for all $r > 0$, the punctured disk $\check{D}_r(a)$ has non-empty intersection with A.

[†] In fact the union of an *arbitrary* collection of open sets (even possibly infinitely many of them) is open. This is not true for closed sets; for example the union of the closed disks $\overline{D}_{1-1/n}(0)$, for $n = 1, 2, 3, \ldots$, is the open unit disk $D = D_1(0)$, which is not closed. Similarly the intersection of an arbitrary collection of closed sets is closed, while the analogous statement for open sets is false in general.

Limits

If $f : A \to \mathbb{C}$ is a complex function, and a and b are complex numbers, we write

$$\lim_{z\to a} f(z) \;=\; b \qquad \text{(or equivalently, } f(z) \to b \text{ as } z \to a)$$

to mean, intuitively, that "$f(z)$ approaches b as z approaches a".[†] More precisely, this means that for every positive real number ε, there should exists a positive real number δ (depending on ε) such that f maps all the points in A that are within a distance δ of a, excluding a itself, to within a distance ε of b. In other words, $\forall\, \varepsilon > 0\,,\ \exists\, \delta > 0$ such that

$$f(A \cap \check{D}_\delta(a)) \;\subset\; D_\varepsilon(b),$$

or equivalently $0 < |z - a| < \delta \Longrightarrow |f(z) - b| < \varepsilon$. Note that a need not be in A, but it is implicitly assumed that a is at least a limit point of A.

Also important is the notion of the limit of a sequence $z_1, z_2, z_3, \ldots$ of complex numbers. We write

$$\lim_{n\to\infty} z_n \;=\; b \qquad \text{(or equivalently, } z_n \to b \text{ as } n \to \infty)$$

to mean $\forall\, \varepsilon > 0\ \exists\, n$ (again depending on ε) such that $z_k \in D_\varepsilon(b)$ for all $k > n$.

The usual limit laws hold: the limit of a sum, difference, product or quotient, is the sum, difference, product or quotient of the limits (provided they exist). You are asked to prove this for sums in the homework.

Continuity

We say that a complex function $f : A \to \mathbb{C}$ is continuous at a point a if it is defined at a (i.e. $a \in A$) and $\lim_{z\to a} f(z) = f(a)$. It is said to be a continuous function if it is continuous at every point in A.

Exercise Show that a complex function is continuous (in the sense defined above) if and only if its real and imaginary parts are continuous (in the sense defined in multivariable calculus). Using this fact, it is easy to show that the functions considered above (exp, log, trig and hyperbolic trig functions, power functions, etc.) are all continuous.

There is an elegant reformulation of continuity for functions whose domains are open:

1.2 Theorem *Let A be open. Then $f : A \to \mathbb{C}$ is continuous $\Longleftrightarrow$ $f^{-1}(U)$ is open for each open subset U of $\mathbb{C}$.*

Proof ($\Longrightarrow$) Let $U \subset \mathbb{C}$ be open. Given $a \in f^{-1}(U)$ (meaning $f(a) \in U$), we must show that $\exists\, \delta > 0$ such that $D_\delta(a) \subset f^{-1}(U)$. To see this, observe that

- U is open and $f(a) \in U \Longrightarrow \exists\, \varepsilon > 0$ such that $D_\varepsilon(f(a)) \subset U$.
- A is open and $a \in A \Longrightarrow \exists\, \delta_1 > 0$ such that $D_{\delta_1}(a) \subset A$
- f is continuous at $a \Longrightarrow \exists\, \delta_2 > 0$ such that $f(A \cap D_{\delta_2}(a)) \subset D_\varepsilon(f(a))$.

Thus taking $\delta = \min(\delta_1, \delta_2)$, we have $f(D_\delta(a)) \subset D_\varepsilon(f(a)) \subset U$, i.e. $D_\delta(a) \subset f^{-1}(U)$.

($\Longleftarrow$) Let $a \in A$ and $\varepsilon > 0$. By hypothesis $f^{-1}(D_\varepsilon(f(a))$ is open, so $\exists\, \delta > 0$ such that

$$D_\delta(a) \;\subset\; f^{-1}(D_\varepsilon(f(a)),$$

that is, $f(D_\delta(a)) \subset D_\varepsilon(f(a))$. Thus f is continuous at a. $\square$

[†] Note that unlike the real case, where z can only approach a from the right or left, in the complex case it can approach a in many different ways, e.g. from the left, right, top or bottom, or perhaps spiraling in.

Remark The theorem holds for general A if the conclusion "$f^{-1}(U)$ is open" is replaced with "$f^{-1}(U)$ is open relative to A" (meaning that $f^{-1}(U)$ is the *intersection* of an open set with A).

Corollary *If f and g are continuous functions, then so is $f \circ g$.*

Proof Apply the theorem twice, noting that $(f \circ g)^{-1}(U) = f^{-1}(g^{-1}(U))$. □

Uniform Continuity

The continuity of $f : A \to \mathbb{C}$, spelled out, means that $\forall\, \varepsilon > 0$ and $a \in A$, $\exists\, \delta > 0$ such that ... The δ may very well depend on both ε and a. If there is a δ that will work for all a, that is, if $\forall\, \varepsilon > 0$, $\exists\, \delta > 0$ such that

$$f(D_\delta(a)) \subset D_\varepsilon(f(a)) \text{ for all } a \in A,$$

or equivalently $|a-b| < \delta \implies |f(a)-f(b)| < \varepsilon$, then we say that f is uniformly continuous. Note that all that has changed is that the quantifiers $\forall\, a$ and $\exists\, \delta$ have been swapped. This is a very important notion in the theoretical study of complex analysis.

Example The function $\check{D}_1(0) \to \mathbb{C}$ that sends z to $1/z$ is continuous but not uniformly continuous; the reader should draw a picture to see why.

We introduce two other important topological notions before moving on:

Connectedness

Definition A subset $A \subset \mathbb{C}$ is path-connected if any two points $z, w \in A$ can be joined by a path in A, meaning a continuous function $\gamma : [a, b] \to A$ (in the sense of multivariable calculus, viewing $A \subset \mathbb{R}^2$) with $\gamma(a) = z$ and $\gamma(b) = w$. If γ can always be chosen to be smooth (i.e. which, at least for now, will mean differentiable), then we say that A is smoothly path-connected.

Remark Slightly more general is the notion of A being connected. This means that the only subsets of A that are clopen (both closed and open) relative to A are A and $\varnothing$. It can be shown that

$$\text{smoothly path-connected} \implies \text{path-connected} \implies \text{connected}$$

while the converses fail. For open sets, however, these notions coincide (exercise).

Definition A region in $\mathbb{C}$ is a subset that is both *open* and *connected*. By the preceding remark, it follows that regions are *smoothly path-connected*. The domains of most of the functions considered in this course will be regions.

It is straightforward to show that ***continuous functions preserve connectedness***, i.e. A connected and $f : A \to \mathbb{C}$ continuous $\implies f(A)$ is connected, and the analogous statement is true for *path-connectedness* (exercise).

Compactness

Definition A subset $A \subset \mathbb{C}$ is compact if every open cover of A (meaning a collection of open sets whose union contains A) has a finite subcover (meaning a finite subcollection of these sets whose union still contains A).

It is straightforward to show that ***continuous functions preserve compactness***, i.e. A compact and $f : A \to \mathbb{C}$ continuous $\implies f(A)$ is compact (exercise).

There is a useful characterization of compact subsets of $\mathbb{C}$ that is often taken as the definition in elementary courses. It uses the notion of a bounded subset of $\mathbb{C}$, meaning a subset that is contained in some disk (possibly of very large, but finite, radius).

1.3 Theorem (Heine-Borel) *$A \subset \mathbb{C}$ is compact if and only if it is closed and bounded.*

This result is proved in every elementary topology course, but we do not prove it here. We do, however, prove the following useful theorem which shows how compactness plays a role in the study of continuous functions:

1.4 Theorem *A continuous complex function f is uniformly continuous on any compact subset A of its domain.*

Proof Let $\varepsilon > 0$. For each $z \in A$, choose $\delta_z > 0$ such that

$$f(D_{\delta_z}(z)) \;\subset\; D_{\varepsilon/2}(f(z)).$$

The collection of all the disks $D_{\delta_z/2}(z)$ for $z \in A$ forms an open cover of A. Since A is compact, A lies in the union of finitely many of these disks, say with centers $z_1, \ldots, z_n$. Set $\delta_k = \delta_{z_k}$ and $\delta = \min(\delta_1/2, \ldots, \delta_n/2)$.

Now consider any pair of points $a, b \in A$ with $|a - b| < \delta$. Certainly a lies in some $D_{\delta_k/2}(z_k)$, and so a fortiori in $D_{\delta_k}(z_k)$. We claim that b also lies in $D_{\delta_k}(z_k)$. Indeed

$$|b - z_k| \;\leqslant\; |b - a| + |a - z_k| \;<\; \delta + \delta_k/2 \;\leqslant\; \delta_k.$$

Therefore $|f(a) - f(b)| \leqslant |f(a) - f(z_k)| + |f(z_k) - f(b)| < \varepsilon/2 + \varepsilon/2 = \varepsilon$. □

D. Differentiability

Definition Let $f : A \to \mathbb{C}$ with A open. We say f is differentiable at a point a in A if

$$\lim_{z \to a} \frac{f(z) - f(a)}{z - a} \qquad \left(\text{or equivalently } \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}\right)$$

exists, and when it does, it is denoted $f'(a)$ or $df/dz\,(a)$ and called the derivative of f at a. Using basic properties of limits, it is easy to show that ***differentiability at a point implies continuity at that point*** (HW).

If f is differentiable at it every point in its domain A, then the resulting function

$$f' \;=\; df/dz : A \;\longrightarrow\; \mathbb{C}$$

is called the derivative of f, and f is said to be differentiable (or synonymously analytic or holomorphic) on A.† A quick exercise with limits shows that the derivative of any constant function is zero, and that $(cf)'(z) = cf'(z)$ for any constant c. In addition, we have:

1.5 Theorem (Rules of Differentiation) *If f, g are analytic on A, B respectively, then*

(a) (sum and difference rules) *$f \pm g$ is analytic on $A \cap B$, and $(f \pm g)' = f' \pm g'$.*

(b) (product rule) *fg is analytic on $A \cap B$, and $(fg)' = f'g + fg'$.*

(c) (quotient rule) *fg is analytic on $A \cap B - g^{-1}(0)$, and $(f/g)' = (f'g - fg')g^2$.*

(d) (chain rule) *$g \circ f$ is analytic on $f^{-1}(B)$, and $(g \circ f)' = (g' \circ f)g'$, i.e.*

$$(g \circ f)'(a) \;=\; g'(f(a))f'(a) \qquad \textit{for all } a \in f^{-1}(B).$$

† Note however that analytic at a means differentiable on some open set containing a.

The proof of (b) is a HW problem, and (a) and (c) are left as exercises. To prove (d), first note that f and g are continuous, since they are differentiable. Setting $w = f(z)$ and $b = f(a)$, we must show

$$(*) \qquad \frac{g(w)-g(b)}{z-a} \longrightarrow g'(b)f'(a) \qquad \text{as } z \longrightarrow a$$

As long as $w \neq b$, the fraction on the left can be written as the product of two fractions, $(g(w)-g(b))/(w-b)$ and $(f(z)-f(a))/(z-a)$, the second of which approaches $f'(a)$ as $z \to a$. But unfortunately w might equal b for z's arbitrarily close to a. To get around this, we introduce a new function $h : B \to \mathbb{C}$, defined by

$$h(w) \;=\; \begin{cases} \dfrac{g(w)-g(b)}{w-b} & \text{if } w \neq b \\ g'(b) & \text{if } w = b. \end{cases}$$

This function is continuous (at b by the defiinition of $g'(b)$, and at all other points in B since g is continuous). Now the fraction on the left in $(*)$ can be rewritten as

$$h(w)\,\frac{f(z)-f(a)}{z-a}.$$

and $h(w) = h(f(z))$ goes to $h(f(a)) = h(b) = g'(b)$ as $z \to a$, as desired. $\square$

Remark The same argument shows that if $\gamma : [a,b] \to A$ is a smooth path and $f : A \to \mathbb{C}$ is analytic, then $f \circ \gamma$ is differentiable with $(f \circ \gamma)'(t) = f'(\gamma(t))\gamma'(t)$.

1.6 Corollary (Zero Derivative Theorem) *If $f : A \to \mathbb{C}$ is analytic and $f'(z) = 0$ for all $z \in A$, then f is constant on any connected open subset U of A.*

Proof Given two points $w, z \in U$, let $\gamma : [0,1] \to U$ be a smooth path with $\gamma(0) = w$ and $\gamma(1) = z$. By the chain rule (in the form of the preceding remark) we have, for all t,

$$(f \circ \gamma)'(t) \;=\; f'(\gamma(t))\gamma'(t) \;=\; 0 \cdot \gamma'(t) \;=\; 0$$

and so $(u \circ \gamma)'(t) = (v \circ \gamma)'(t) = 0$ where u and v are the real and imaginary parts of f. From calculus (in particular the mean value theorem) it follows that u and v are constant functions of t, and so f is as well. Therefore $f(w) = f(z)$. Since w and z were arbitrary, it follows that f is constant on U. $\square$

At the opposite extreme, if $f'(z) \neq 0$ for all $z \in A$, then f is "angle preserving" (or "conformal") in the following sense:

Definition A function $f : A \to \mathbb{C}$ is said to be conformal at $z \in A$ if there exists an angle $\theta \in [0, 2\pi)$ and a scalar $r > 0$ such that near z, the map f (infinitesimally) rotates by θ and dilates by r. More precisely, for every curve $\gamma : \mathbb{R} \to A$ satisfying $\gamma(0) = z$ and $\gamma'(0) \neq 0$, the image curve $\mu := f \circ \gamma$ is differentiable at 0 with $\mu'(0) \neq 0$, and

$$|\mu'(0)| \;=\; r|\gamma'(0)| \qquad \text{and} \qquad \arg(\mu'(0)) = \arg(\gamma'(0)) + \theta.$$

In particular, f preserves angles between intersecting curves.

1.7 Corollary (Conformal Mapping Theorem) *If f is analytic at z and $f'(z) \neq 0$, then f is conformal at z with $\theta = \arg f'(z)$ and $r = |f'(z)|$ (in the definition above).*

Proof For any smooth curve γ through z as in the definition, with image curve $\mu = f \circ \gamma$, we have $\mu'(0) = f'(z)\gamma'(0)$, by the chain rule. The result follows by the analysis of multiplication by $f'(z)$ as discussed on page 3. $\square$

The Cauchy-Riemann (CR) Equations

These fundamental equations characterize the analyticity of a complex function in terms of the partial derivatives of its real and imaginary parts.

1.8 Cauchy–Riemann Theorem *A complex function $f = u + iv$ is analytic if and only if it is differentiable as a real function and satisfies the Cauchy–Riemann Equations*

$$u_x \;=\; v_y \qquad \text{and} \qquad u_y \;=\; -v_x$$

where subscripts denote partial derivatives. In this case $f' = f_x = u_x + iv_x = f_y = v_y - iu_y$.

Before giving the proof, we recall that a real function $f : A \to \mathbb{R}^2$, with $A \subset \mathbb{R}^2$ open, is said to be differentiable if it is differentiable at each point $a \in A$, where differentiability at a means that there exists a linear map $T : \mathbb{R}^2 \to \mathbb{R}^2$ such that

$$\lim_{h\to 0} \frac{f(a+h) - f(a) - T(h)}{|h|} \;=\; 0 \qquad (\text{note that } h \in \mathbb{R}^2)$$

(cf. Chapter 2 in Spivak's *Calculus on Manifolds*). It is easy to show that T is unique if it exists.† It is typically denoted df_a, and is called the derivative (or differential) of f at a.

Fact (See Spivak, for example) If f is differentiable with components u and v (i.e. $f(x,y) = (u(x,y), v(x,y))$) then all the partial derivatives u_x, v_x, u_y, v_y exist, and df_a (for each a) is represented with respect to the standard basis of $\mathbb{R}^2$ by the Jacobian matrix

$$Jf_a \;=\; \begin{pmatrix} u_x(a) & u_y(a) \\ v_x(a) & v_y(a) \end{pmatrix} \qquad \left(\text{or just write } Jf \;=\; \begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix}\right).$$

Conversely, if all the partials of f exist and are continuous, then f is differentiable.

Note: ***The CR equations just say that Jf is an "amplitwist" matrix, meaning its second column is its first rotated a quarter turn counterclockwise.***

Proof (of the Cauchy-Riemann Theorem) Suppose f is analytic at z with $f'(z) = a + ib$. This means that $\lim_{h\to 0}(f(z+h) - f(z))/h = a + ib$, or equivalently

$$\lim_{h\to 0} \frac{f(z+h) - f(z) - (a+ib)h}{|h|} \;=\; \lim_{h\to 0} \frac{f(z+h) - f(z) - \begin{pmatrix} a & -b \\ b & a \end{pmatrix} h}{|h|} \;=\; 0$$

where we have identified $\mathbb{C}$ with $\mathbb{R}^2$ in the second limit. Thus f is differentiable as a real function with

$$Jf_z \;=\; \begin{pmatrix} a & -b \\ b & a \end{pmatrix} \;=\; \begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix}$$

and so $u_x \;=\; v_y$ and $v_x \;=\; -u_y$.

Conversely, if f is differentiable as a real function and satisfies the CR equations, then

$$df_z \;=\; \begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix} \;=\; \begin{pmatrix} u_x & -v_x \\ v_x & u_x \end{pmatrix}$$

so viewing f as a complex function we have

$$\lim_{h\to 0} \frac{f(z+h) - f(z) - (u_x + iv_x)h}{h} \;\longrightarrow\; 0$$

so $f'(z)$ exists and equals $u_x + iv_x = v_y - iu_y$. $\square$

† If S is another such linear map, then $\lim_{h\to 0}((S(h) - T(h))/|h|) = 0$ (by taking the difference of the defining limits for T and S). Replacing h by tu, where $u = h/|h|$ and t is real, and letting $t \to 0$, we see (using the linearity of S and T) that $S(u) = T(u)$. Thus $S = T$ on all unit vectors, and so $S = T$.

Applications (derivatives of exp, trig functions, log and power functions)

① Recall that $e^z = u + iv$ where (for $z = x + iy$) $u = e^x \cos y$ and $v = e^x \sin y$. The functions u and v are continuously differentiable with

$$u_x = e^x \cos y = v_y \qquad \text{and} \qquad u_y = -e^x \sin y = -v_x.$$

Thus e^z is analytic by the CR Theorem, with

$$de^z/dz = u_x + iv_x = e^x(\cos y + i \sin y) = e^z.$$

② Using the rules of differentiation, we deduce from ① that the sine function is analytic, with $\sin' = \cos$: By the chain rule $de^{\pm iz}/dz = \pm ie^{iz}$, so by 1.5(a), sin is analytic with

$$\sin' z = \frac{ie^{iz} - (-ie^{-iz})}{2i} = \frac{e^{iz} + e^{-iz}}{2} = \cos z.$$

Similarly $\cos' = -\sin$, $\tan' = \sec^2$ (where $\sec = 1/\cos$), $\sinh' = \cosh$, $\cosh' = \sinh$, etc.

③ The logarithm (any branch) is analytic at any $z \neq 0$, with

$$\log'(z) = 1/z.$$

There are several (instructive) ways to see this.

ⓐ For example, away from the imaginary axis (where $x = 0$) we have $\log = u + iv$ where $u = \log{(x^2 + y^2)}^{1/2}$ and $v = \arctan(y/x)$ (for some branch of arctan), which are continuously differentiable with

$$u_x = \frac{x}{x^2 + y^2} = \frac{1/x}{1 + (y/x)^2} = v_y \quad \text{and} \quad u_y = \frac{y}{x^2 + y^2} = -\frac{-y/x^2}{1 + (y/x)^2} = -v_x$$

and so the CR Theorem gives the result in this region.

ⓑ To give a proof for all $z = re^{i\theta} \neq 0$, one can use the Polar Cauchy-Riemann Equations[†]

$$u_r = \frac{1}{r}\, v_\theta \qquad \text{and} \qquad v_r = -\frac{1}{r}\, u_\theta$$

(also useful in physics). Since $\log z = u + iv$ where $u = \log r$ and $v = \theta$, we check that

$$u_r = 1/r = v_\theta/r \qquad \text{and} \qquad v_r = 0 = -u_\theta/r$$

so log is analytic. We can now compute $\log'(z)$ by approaching $z = re^{i\theta}$ in any manner, e.g. along the ray $(r + t)e^{i\theta}$ as $t \to 0$:

$$\log'(z) = \lim_{t\to 0} \frac{(\log(r + t) + i\theta) - (\log r + i\theta)}{te^{i\theta}} = \frac{1}{e^{i\theta}} \log'(r) = \frac{1}{re^{i\theta}} = \frac{1}{z}.$$

(Note: MH's proof on page 83 is flawed because $\arctan(y/x)$ is undefined when $x = 0$.)

ⓒ One can also compute $\log'$ using the Inverse Function Theorem (see below).

④ For b nonzero, $dz^b/dz = de^{b \log z}/dz = bz^b/b$ (by the chain rule) which equals bz^{b-1}.

[†] These follow from the usual CR equations using the change of variables $x = r\cos\theta$, $y = r\sin\theta$: The chain rule gives

$$\begin{pmatrix} u_r & u_\theta \\ v_r & v_\theta \end{pmatrix} = \begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix} \begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{pmatrix}$$

or equivalently

$$\begin{pmatrix} u_r & u_\theta/r \\ v_r & v_\theta/r \end{pmatrix} = \begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix} \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}.$$

and the polar CR equations follow, since the product of an amplitwist matrix and a rotation matrix is an amplitwist matrix.

For another application of the CR equations, observe that ***if $f = u + iv$ is analytic, then the level sets of u and v are orthogonal***. Indeed, from calculus we know that these sets are perpendicular to the gradient vector fields of u and v, so it suffices to show

$$(u_x, u_y) \cdot (v_x, v_y) \;=\; u_x v_x + u_y v_y \;=\; 0$$

which is immediate from the CR equations. As an example, any branch of the function arcsin is analytic (by the inverse function theorem below) and so this shows that the ellipses and hyperbolas drawn in Figure 5 are orthogonal.

We conclude this section with two more important results about analytic functions that can be proved using the CR Theorem :

1.9 Inverse Function Theorem *If f is analytic at z* (meaning differentiable near z) *with $f'(z) \neq 0$, then there exist open sets U containing z, and V containing $w = f(z)$, such that $f : U \to V$ is bijective and $f^{-1} : V \to U$ is analytic with $(f^{-1})'(w) = 1/f'(z)$.*

This is rather tricky to prove. One approach (taken in MH) is to appeal to the real version of this theorem, where the hypothesis $f'(z) \neq 0$ is replaced with $\det(df_z) \neq 0$, and the conclusion $(f^{-1})'(w) = 1/f'(z)$ is replaced with $(df^{-1})_w = df_z^{-1}$. Then one only need observe that the amplitwist matrix representing multiplication by $f'(z)$ has nonzero determinant (namely $|f'(z)|$) and that its inverse is also an amplitwist matrix, and then appeal to the CR Theorem. But of course one must still prove the real inverse function theorem, which is hard. We do not give the proof here.

Applications 1 Using the fact that log and exp are inverse functions, this theorem gives an alternative proof to the one above that (any branch of) log is analytic, and that $\log'(z) = 1/\exp'(\log z) = 1/\exp(\log z) = 1/z$.

2 If f is analytic with $f' \neq 0$ everywhere in some open set A, then $f(A)$ is open. Indeed we can assume that the open sets U in the theorem lie in A, and so $f(A)$ is the union of all the corresponding open sets V, and so is open.†

3 Combining this theorem with Corollary 1.6 (the zero derivative theorem) it is easy to prove that if f is analytic on a connected open set A with constant modulus (meaning $|f|$ is constant on A), then f is constant on A. The proof is asked for in the homework.

Harmonic functions Let A be an open set in $\mathbb{R}^2$. A twice continuously differentiable function $h : A \to \mathbb{R}$ (a.k.a. a C^2-function) is harmonic if

$$\Delta h \;:=\; h_{xx} + h_{yy} \;=\; 0$$

at every point in A. Here the double subscripts indicate second order partial derivatives (so $h_{xx} = \partial^2 h/\partial x^2$, etc.). The differential operator Δ is called the Laplacian, and is one of the most important operators in mathematics and physics.

1.10 Theorem *If $f = u + iv$ is analytic, then u and v are harmonic. Conversely, if u is harmonic on an open disk D, then there exists a harmonic function v on D, unique up to adding a constant, such that $u + iv$ is analytic; we call v the harmonic conjugate of u.*

Proof The first statement follows from the CR equations: $u_{xx} = v_{yx}$ and $u_{yy} = -v_{xy}$, and these add up to zero by the equality of mixed partials (from in multivariable calculus). Similarly $v_{xx} + v_{yy} = 0$. The second statement follows from the basic existence and uniqueness theorems for differential equations, but we do not prove it here. □

† This is a special case of the "Open Mapping Theorem" (to be proved later) that states that any nonconstant analytic function on a connected open set is an "open map", i.e. maps open sets to open sets.

***Remark** (starred sections are optional for undergraduates)

The conjugate v of a given harmonic function u can be found by integration. First compute $w = \int -u_y\,dx$ (with a fixed constant of integration) and set $h = u_x - w_y$, which is a function of y only since $h_x = u_{xx} - w_{yx} = u_{xx} + u_{yy} = 0$. Then

$$v \;=\; w - \int h(y)dy.$$

To see that that $f = u + iv$ is analytic, compute $v_x = w_x = -u_y$ and $v_y = w_y + h = u_x$, which are the Cauchy-Riemann equations.

Note that v is defined up to adding a *real* constant, or equivalently, f is defined up to adding a *purely imaginary* constant.

Ahlfors describes a simpler way to find f, without integrating. First assume that u is defined at $(0,0)$. Then

$$f(z) \;=\; u(z/2, z/2i) - u(0,0).$$

This formula is derived using the following magic. Set $z = x + iy$ and $\bar{z} = x - iy$. If $f(z) = u(x,y) + iv(x,y)$ is analytic, then setting $\bar{f}(\bar{z}) = u(x,y) - iv(x,y)$ we have

$$u(x,y) \;=\; \tfrac{1}{2}(f(z) + \bar{f}(\bar{z})).$$

It is reasonable to assume (and can in fact be shown) that this last identity holds for all complex x and y. Taking $x = z/2$ and $y = z/2i$, and so $z = z/2 + i(z/2i) = x + iy$ and $\bar{z} = x - iy = z/2 - i(z/2i) = 0$, we find that $u(z/2, z/2i) = \frac{1}{2}(f(z) + \bar{f}(0))$. Since f can be changed by adding an imaginary constant, we may assume that $f(0)$ is real, and so $\bar{f}(0) = u(0,0)$. This gives the stated formula.

If u is not defined at $(0,0)$, then $u_0 = u \circ \tau$ is, for a suitable translation τ of $\mathbb{R}^2$. By the argument above there is an analytic f_0 with real part u_0. Then u is the real part of the analytic function $f = f_0 \circ \tau^{-1}$, where τ is now viewed as a translation of $\mathbb{C}$.

2. Integration

A. Contour Integrals

Definition A contour is a *smooth* map $\gamma : [a,b] \to \mathbb{C}$, meaning that γ' exists and is continuous and nonzero on (a,b), and that the one-sided limits of γ' exist at the endpoints a and b. There is a natural *orientation* on the image *curve* $C = \mathrm{Im}(\gamma)$, from $z = \gamma(a)$ toward $w = \gamma(b)$, indicated by putting an arrow on C; we say that γ is a contour *from z to w*. If $z = w$, then γ is called a closed contour. If $\gamma(s) \neq \gamma(t)$ except when $s = t$ (or possibly $s, t = a, b$ in some order; i.e. C does not "intersect itself") then γ is called a simple contour.

Remark We often blur the distinction between the *map* γ and the *oriented curve* C, and talk about "the curve γ" to mean C. Strictly speaking, γ is just one of infinitely many possible parametrizations of C. For example any map $\gamma \circ \rho$, where $\rho : [c,d] \to [a,b]$ is smooth with $\rho(c) = a$ and $\rho(d) = b$, is a reparametrization of C.

Given a contour $\gamma : [a,b] \to \mathbb{C}$ and a function f that is continuous on γ, define

$$\int_\gamma f \;=\; \int_\gamma f(z)\,dz \;:=\; \int_a^b f(\gamma(t))\gamma'(t)\,dt.^{\dagger}$$

† Note that the integrand is complex, and is computed in terms of real integrals by defining $\int(u(t) + iv(t))dt := \int u(t)\,dt + i\int v(t)\,dt$.

An easy calculus exercise shows that the result is independent of the parametrization of $C = \mathrm{Im}(\gamma)$, and so we can write $\int_C$ in place of $\int_\gamma$. If C is closed, we sometimes write $\oint_C$, or even just $\oint$ if C is understood from the context. The reader should compare this definition of contour integrals with definition of line integrals of vector fields, where complex multiplication is replaced by the dot product. It is also instructive to write down the definition as a limit of Riemann sums.

Examples ① Let S be the oriented line segment in the complex plane from 1 to i, parametrized by $\sigma(t) = 1 + t(i-1) = (1-t) + it$ for $t \in [0,1]$. Then

$$\begin{aligned}\int_S z\,dz &= \int_\sigma z\,dz = \int_0^1 ((1-t)+it)(-1+i)dt \\ &= \int_0^1 (-1+i(1-2t))dt = (-t+i(t-t^2))\Big|_0^1 = -1.\end{aligned}$$

② Let C be the counterclockwise oriented unit circle, parametrized by $\tau(t) = e^{it}$ for $t \in [0, 2\pi]$. Then

$$\oint_C \frac{1}{z}\,dz = \int_\tau \frac{1}{z}\,dz = \int_0^{2\pi} \frac{1}{e^{it}} ie^{it}\,dt = it\Big|_0^{2\pi} = 2\pi i.$$

Remark It is sometimes convenient to use *piecewise smooth* maps $\gamma : [a,b] \to \mathbb{C}$ to parametrize a curve C (especially if C has corners), meaning γ is smooth on each subinterval $[a_k, a_{k+1}]$ of a partition $a = a_1 < \cdots < a_n = b$ of $[a,b]$; the one-sided limits of γ' need not agree at $a_1, \ldots, a_n$. We will continue to call such maps "contours", and define

$$\int_C f = \int_\gamma f := \sum_{k=1}^{n-1} \left(\int_{\gamma_k} f \right)$$

where $\gamma_k = \gamma | [a_k, a_{k+1}]$.

2.1 Fundamental Theorem of Contour Integrals *If f is continuous in a region A and has an antiderivative F there (meaning F is analytic on A and $F' = f$) and γ is a contour in A (meaning its image lies in A) from z to w, then*

$$\int_\gamma f = F(w) - F(z).$$

Note : This theorem shows that $\int_\gamma f$ is the same for *any* contour γ from z to w in A, provided f has an antiderivative (a.k.a. a primitive) throughout A.

Proof First assume γ is smooth.

$$\begin{aligned}\int_\gamma f &= \int_a^b f(\gamma(t)\gamma'(t)\,dt = \int_a^b F'(\gamma(t))\gamma'(t)\,dt \\ &= \int_a^b (F\circ\gamma)'(t)\,dt \underset{\text{FTC}}{=} (F\circ\gamma)(b) - (F\circ\gamma)(a) = F(z_2) - F(z_1).\end{aligned}$$

In the piecewise smooth case, one obtains a sum that telescopes. □

Examples (from above, again) ① $\displaystyle\int_\sigma z = \frac{1}{2}z^2\Big|_1^i = -\frac{1}{2} - \frac{1}{2} = -1.$

② We can't use 2.1 to compute $\oint \frac{1}{z}\,dz$ since log is not single valued on the unit circle.

2.2 **Estimation Theorem** *If $\gamma : [a,b] \to \mathbb{C}$ is a contour of length $L := \int_a^b |\gamma'(t)|dt$ and f is a continuous function on γ with $|f(z)| \leqslant M$ for all z on γ, then*

$$\left|\int_\gamma f\right| \;\leqslant\; ML.$$

Proof By definition, $\left|\int_\gamma f\right| \;=\; \left|\int_a^b f(\gamma(t))\gamma'(t)\,dt\right| \;\leqslant\; \int_a^b |f(\gamma(t))\gamma'(t)|dt$ where the inequality follows from the general fact that

$$\left|\int_a^b g(t)\,dt\right| \;\leqslant\; \int_a^b |g(t)|dt.$$

for any continuous function $g : [a,b] \to \mathbb{C}$.† Since the norm is multiplicative, we have

$$\left|\int_\gamma f\right| \;\leqslant\; \int_a^b |f(\gamma(t))|\,|\gamma'(t)|\,dt \;\leqslant\; M\int_a^b |\gamma'(t)|dt \;=\; ML. \quad \square$$

B. Cauchy's Theorem

Classical statement: If f is *analytic on and inside* a simple closed contour γ, then

$$\int_\gamma f(z)\,dz \;=\; 0.$$

But what does "inside" mean? We take a more modern approach.

Homotopic contours

Let z and w be two points in a region $A \subset \mathbb{C}$. Two contours γ, $\delta : [a,b] \to A$ from z to w are homotopic (written $\gamma \simeq \delta$) in A if there exists a one-parameter family of contours $\gamma_s : [a,b] \to A$, $s \in [0,1]$, all from z to w, such that $\gamma_0 = \gamma$, $\gamma_1 = \delta$, and the function

$$H : [a,b] \times [0,1] \;\longrightarrow\; A \qquad (s,t) \;\longmapsto\; \gamma_s(t)$$

is continuous. Such a map H is called a homotopy from γ to δ.

If $z = w$ (so γ is closed) then γ is null-homotopic in A if $\gamma \simeq \star$ where $\star$ denotes the constant contour at z, i.e. $\star(t) = z$ for all t. If *every* closed contour in A is null-homotopic, then we say that A is simply connected. Intuitively, this means that every closed loop in A can be shrunk in A to a point.

2.3 **Cauchy's Theorem** *If f is analytic on a simply connected region A, then*

$$\int_\gamma f(z)\,dz \;=\; 0$$

for any closed contour γ in A.

† This is well known if g is real valued. For g complex valued, suppose $\int_a^b g(t)\,dt = re^{i\theta}$. Then

$$\begin{aligned}\left|\int_a^b g(t)\,dt\right| &= r \;=\; \mathrm{Re}(r) \;=\; \mathrm{Re}\left(e^{-i\theta}\int_a^b g(t)\,dt\right) \;=\; \mathrm{Re}\left(\int_a^b e^{-i\theta}g(t)\,dt\right) \\ &= \int_a^b \mathrm{Re}\left(e^{i\theta}g(t)\right)dt \;\leqslant\; \int_a^b \left|e^{i\theta}g(t)\right|dt \;=\; \int_a^b |g(t)|dt.\end{aligned}$$

Remark The same conclusion holds if we allow there to be one point $a \in A$ where f, though still continuous, need not be assumed to be differentiable. We refer to this as the generalized Cauchy's Theorem.

Below we will give the classic proof due to Goursat in 1883 for the case when A is a disk, and then use this special case to prove the following powerful generalization:

2.4 Deformation Theorem *If f is analytic on an arbitrary region A, and γ and δ are homotopic contours in A, then*

$$\int_\gamma f(z)\,dz \;=\; \int_\delta f(z)\,dz.$$

In particular $\int_\gamma f = 0$ for any null-homotopic closed contour in A.

Note that 2.3 follows immediately from the last statement in 2.4, which in turn follows from the first statement since $\gamma \simeq \star$ implies that $\int_\gamma f = \int_\star f = 0$ (since $\star'(t) = 0$ for all t).

The deformation theorem is often used to simplify the computation of contour integrals by replacing the contour with a "simpler" one homotopic to γ. For example, if γ is any contour "encircling" the origin once counterclockwise, then

$$\int_\gamma \frac{1}{z}\,dz \;=\; 2\pi i$$

since γ is homotopic to a counterclockwise circle C centered at the origin, and it is easy to show $\int_C dz/z = 2\pi i$ as in example 2 above. Other examples appear in the homework.

Here is one very useful application of Cauchy's Theorem:

2.5 Primitive Theorem *If f is analytic on a simply connected region A, then f has a primitive F (i.e. $F' = f$) on A, unique up to adding a constant.*

Proof Pick $a \in A$ and set

$$F(z) \;=\; \int_a^z f(z)\,dz$$

where $\int_a^z$ means $\int_\gamma$ for any contour in A from a to z. This is well defined by Cauchy.[†]

We claim $F' = f$. Using any contour to get from a to a given z, and extending this by a straight line from there to $z+h$ (for suitably small h), we see that

$$F(z+h) - F(z) \;=\; \int_0^1 f(z+th)h\,dt.$$

Therefore, using the estimation theorem we compute

$$\left|\frac{F(z+h)-F(z)}{h} - f(z)\right| \;=\; \left|\frac{\int_0^1 (f(z+th)-f(z))h\,dt}{h}\right| \;\leqslant\; \max_{t\in[0,1]} \frac{|f(z+th)-f(z)||h|}{|h|}$$

which tends to 0 as $h \to 0$, by the continuity of f at z, and so $F'(z) = f(z)$.

If $G' = f$ as well, then $(F-G)' = 0 \implies F - G$ is constant since A is connected. $\square$

[†] If δ is another such path, then $\gamma - \delta$, meaning traverse γ from a to z and then δ "backwards" from z back to a, is a closed path in A, and so $\int_{\gamma-\delta} f = 0$. But this is equal to $\int_\gamma f - \int_\delta f$, so $\int_\gamma f = \int_\delta f$.

2.6 **Corollary** *Let A be a simply connected region not containing* 0*. Then there exists a continuous function $\ell : A \to \mathbb{C}$, unique up to addition of multiples of $2\pi i$, such that $e^{\ell(z)} = z$. (We call ℓ a generalized branch of the logarithm.)*

Proof By the theorem, $\exists\ p : A \to \mathbb{C}$ with $p'(z) = 1/z$. Fix any $a \in A$, and set

$$\ell(z) \;=\; p(z) - p(a) + \log(a)$$

for any branch of log defined at a. Then

$$\ell'(z) = 1/z \qquad \text{and} \qquad e^{\ell(a)} = e^{\log(a)} = a.$$

In fact $e^{\ell(z)} = z$ for all $z \in A$. To see this, set $q(z) = e^{\ell(z)}/z$. Then $q'(z) = 0$ by the quotient rule, so q is constant. Therefore $q(z) = q(a) = 1$, and so $e^{\ell(z)} = z$. $\square$

Before giving the promised proof of Cauchy's Theorem (and the Deformation Theorem) we discuss some other remarkable consequences.

C. Cauchy's Integral Formula

If f is analytic at z, then Cauchy's integral formula expresses the value of f at z in terms of the values of f on any closed curve "encircling" z; it is remarkable that this is possible, underscoring the rigidity of analytic functions.

To state this formula precisely, consider a closed contour $\gamma : [a,b] \to \mathbb{C}$ and a point z not on γ. Define the index (or winding number) of γ about z to be

$$\mathrm{I}(\gamma, z) \;:=\; \frac{1}{2\pi i}\int_\gamma \frac{1}{\zeta - z}\, d\zeta\,.$$

This definition (motivated by our previous computation $\mathrm{I}(\gamma, 0) = 1$ when γ is the counterclockwise unit circle centered at the origin) is reasonable in view of the fact that $\mathbf{I}(\boldsymbol{\gamma}, \boldsymbol{z})$ ***is always an integer***. Indeed $\mathrm{I}(\gamma, z) = g(b)/2\pi i$, where

$$g(s) \;=\; \int_{\gamma|[a,s]} \frac{1}{\zeta - z}\, d\zeta \;=\; \int_a^s \frac{\gamma'(t)}{\gamma(t) - z}\, dt\,.$$

By the Fundamental Theorem of Calculus, $g'(s) = \gamma'(s)/(\gamma(s) - z)$, or equivalently $\gamma'(s) - g'(s)(\gamma(s) - z) = 0$. This implies that $h'(s) = 0$ where

$$h(s) \;=\; e^{-g(s)}(\gamma(s) - z)\,.$$

Thus h is constant, so $h(a) = h(b)$. Since $\gamma(a) = \gamma(b)$, it follows that $e^{-g(b)} = e^{-g(a)} = e^{-0} = 1$. Therefore $g(b) = 2\pi i n$ for some $n \in \mathbb{Z}$, so $\mathrm{I}(\gamma, z) = n$, as claimed. $\square$

2.7 **Cauchy's Integral Formula** *Let $f : A \to \mathbb{C}$ be analytic. Then for any $z \in A$ and any null-homotopic closed contour γ in A that does not pass through z,*

$$\mathrm{I}(\gamma, z)\, f(z) \;=\; \frac{1}{2\pi i}\int_\gamma \frac{f(\zeta)}{\zeta - z}\, d\zeta\,.$$

Proof Set

$$g(\zeta) \;=\; \begin{cases} \dfrac{f(\zeta) - f(z)}{\zeta - z} & \text{if } \zeta \neq z \\[2ex] f'(z) & \text{if } \zeta = z \end{cases}$$

which is continuous on A and analytic on $A - \{z\}$. By the generalized Cauchy Theorem

$$0 \;=\; \int_\gamma g \;=\; \int_\gamma \frac{f(\zeta)}{\zeta - z}\, d\zeta - \int_\gamma \frac{f(z)}{\zeta - z}\, d\zeta \;=\; \int_\gamma \frac{f(\zeta)}{\zeta - z}\, d\zeta - 2\pi i f(z)\mathrm{I}(\gamma, z)$$

and the formula follows. $\square$

2.8 **Cauchy's Derivative Formula** *Let $f : A \to \mathbb{C}$ be analytic. Then all derivatives $f^{(k)}$ for $k = 1, 2, \ldots$ exist, and for any z and γ as in 2.7,*

$$\mathrm{I}(\gamma, z)\, f^{(k)}(z) \;=\; \frac{k!}{2\pi i} \int_\gamma \frac{f(\zeta)}{(\zeta - z)^{k+1}}\, d\zeta\, .$$

This follows immediately from the integral formula 2.7 and the following technical result that allows one to differentiate under the integral sign:

2.9 **Interchange Lemma** *Let $g(z, \zeta)$ be a continuous function of z and ζ for z in an open set A and ζ on a contour C. If g is analytic in z for each fixed ζ, then*

$$\frac{d}{dz} \int_C g(z, \zeta)\, d\zeta \;=\; \int_C \frac{\partial g}{\partial z}(z, \zeta)\, d\zeta\, .$$

Proof Fix $z \in A$ and let D be an open disk in A containing z. Then for each ζ on C,

$$g(z, \zeta) \;=\; \frac{1}{2\pi i} \int_{\partial D} \frac{g(\tau, \zeta)}{\tau - z}\, d\tau$$

by Cauchy's integral formula applied to $g(\cdot, \zeta)$. Setting $G(z) = \displaystyle\int_C g(z, \zeta)\, d\zeta$, we have

$$\begin{aligned} G(z) \;&=\; \frac{1}{2\pi i} \int_C \int_{\partial D} \frac{g(\tau, \zeta)}{\tau - z}\, d\tau\, d\zeta \\ &=\; \frac{1}{2\pi i} \int_{\partial D} \int_C \frac{g(\tau, \zeta)}{\tau - z}\, d\zeta\, d\tau \qquad \text{(by Fubini's theorem)} \\ &=\; \frac{1}{2\pi i} \int_{\partial D} \frac{G(\tau)}{\tau - z}\, d\tau \end{aligned}$$

and so G also satisfies Cauchy's integral formula.

Once we know this about G, it follows that G is analytic at z with

$$(*) \qquad G'(z) \;=\; \frac{1}{2\pi i} \int_{\partial D} \frac{G(\tau)}{(\tau - z)^2}\, d\tau\, .$$

(which is first case of the derivative formula for G). To see this, let $r = \mathrm{radius}(D)$ and $M = \max_{\tau \in \partial D} |G(\tau)|$. Then for h nonzero with $|h| < r/2$,we compute, using Cauchy's integral formula for G (established above) and the Estimation Theorem,

$$\begin{aligned} &\left| \frac{G(z+h) - G(z)}{h} - \frac{1}{2\pi i} \int_{\partial D} \frac{G(\tau)}{(\tau - z)^2}\, d\tau \right| \\ &\quad=\; \frac{1}{2\pi} \left| \int_{\partial D} \frac{G(\tau)}{h} \left(\frac{1}{\tau - (z+h)} - \frac{1}{\tau - z} - \frac{h}{(\tau - z)^2} \right) d\tau \right| \\ &\quad=\; \frac{1}{2\pi} \left| \int_{\partial D} \frac{G(\tau)h}{(\tau - (z+h))(\tau - z)^2}\, d\tau \right| \;\leqslant\; \frac{1}{2\pi} 2\pi r \frac{M|h|}{(r/2)r^2} \;=\; \frac{2M}{r^2}|h| \end{aligned}$$

which goes to 0 as $h \to 0$. This proves $(*)$.

Thus

$$\begin{aligned}
\frac{d}{dz}\int_C g(z,\zeta)\,d\zeta \;&=\; G'(z) \underset{(*)}{=} \frac{1}{2\pi i}\int_{\partial D}\int_C \frac{g(\tau,\zeta)}{(\tau-z)^2}\,d\zeta\,d\tau \\
&=\; \frac{1}{2\pi i}\int_C\int_{\partial D} \frac{g(\tau,\zeta)}{(\tau-z)^2}\,d\tau\,d\zeta \\
&=\; \int_C \left(\frac{1}{2\pi i}\int_{\partial D} \frac{g(\tau,\zeta)}{(\tau-z)^2}\,d\tau\right) d\zeta \;=\; \int_C \frac{\partial g}{\partial z}(z,\zeta)\,d\zeta\,.
\end{aligned}$$

where the last equality follows from $(*)$ applied to $g(\cdot,\zeta)$. $\square$

D. Consequences of Cauchy's Formulas

From the Integral Formula

2.10 **Mean Value Property** **(MVP)** *If f is analytic on a region containing a closed disk D centered at a point z, then $f(z)$ is the average of the values of f on ∂D, meaning*

$$f(z) \;=\; \frac{1}{2\pi}\int_0^{2\pi} f(z+re^{i\theta})\,d\theta$$

where r is the radius of D.

Proof By Cauchy's Integral Formula, we have $f(z) = \dfrac{1}{2\pi i}\displaystyle\int_{\partial D}\frac{f(\zeta)}{\zeta - z}\,d\zeta$.

Parametrizing ∂D by $z + re^{i\theta}$ for $\theta \in [0, 2\pi]$ gives

$$f(z) \;=\; \frac{1}{2\pi i}\int_0^{2\pi} \frac{f(z+re^{i\theta})}{re^{i\theta}}\,rie^{i\theta}\,d\theta$$

and the result follows by canceling $rie^{i\theta}$ from numerator and denominator. $\square$

2.11 **Maximum Principle** *Let A be a region with compact closure $\bar{A}$ (the closure of A is A together with all its limit points) and $f : \bar{A} \to \mathbb{C}$ be a continuous, non-constant function that is analytic on A. Then f assumes its maximum modulus only at points on the boundary $\partial A := \bar{A} - A$.*

Proof Let $M = \max_{z\in\bar{A}} |f(z)|$ and $B \;=\; \{z \in A \,:\, |f(z)| = M\}$. We must show $B = \varnothing$.

First note that B is closed in A, since $B = f^{-1}(M) \cap A$, and $f^{-1}(M)$ is closed in $\mathbb{C}$ since f is continuous. We claim that B is open as well. If not, then we could find a disk $D \subset A$, say of radius r, centered at a point $z \in B$ but having points in ∂D that are not in B. But then by the Mean Value Property,

$$|f(z)| \;=\; \left|\frac{1}{2\pi}\int_0^{2\pi} f(z+re^{i\theta})\,d\theta\right| \;\leqslant\; \frac{1}{2\pi}\int_0^{2\pi} |f(z+re^{i\theta}|\,d\theta \;<\; \frac{1}{2\pi}\,2\pi M \;=\; M$$

a contradiction, since $|f(z)| = M$. Thus B is both open and closed in A.

Since A is connected, $B = \varnothing$ or A. But if $B = A$, then $|f|$ would be constant on A, which would imply (e.g. by previous homework) that f is constant. Therefore $B = \varnothing$. $\square$

*Application of the Maximum Principle

The Maximum Principle can be applied to study analytic functions on the open unit disc $D = D_1(0)$.

2.12 Schwarz Lemma *If $f : D \to D$ is analytic with $f(0) = 0$, then*

$$|f(z)| \leqslant |z| \quad \textit{and} \quad |f'(0)| \leqslant 1.$$

If either $|f(z| = |z|$ for some $z \neq 0$ or $|f'(0)| = 1$, then f is a rotation (i.e. $f(z) = uz$ for some unit complex number u). In particular, if $f : D \to D$ is analytic with $f(0) = 0$ and $f'(0) = 1$, then f is the identity function.

Proof Apply the Maximum Principle to the function

$$g(z) \;=\; \begin{cases} f(z)/z & \text{if } z \neq 0 \\ f'(0) & \text{if } z = 0 \end{cases}$$

restricted to the discs $D_r(0)$ as $r \to 1$. The details are left to the reader. $\square$

Remark The Maximum Principle fails for unbounded regions, e.g. for A the infinite horizontal strip $\{z : \operatorname{Im}(z) \in (-\pi/2, \pi/2)\}$. Indeed the function

$$f : \bar{A} \longrightarrow \mathbb{C} \quad \text{given by} \quad f(z) = e^{(e^z)}$$

is bounded on ∂A but unbounded on A (exercise). Under suitable restrictions on f (for example requiring that f be bounded on ∂A and not grow "too fast" on A) one can still conclude that f assumes its maximum modulus on ∂A; these variations on the Maximum Principle go under the name Phragmén-Lindelöf Principles.

From the Derivative Formula

2.13 Morera's Theorem (Converse of Cauchy's Theorem) *If f is continuous on a region A, and $\int_\gamma f = 0$ for all closed contours in A, then f is analytic on A.*

Proof Fix $a \in A$ and define $F : A \to \mathbb{C}$ by

$$F(z) \;=\; \int_a^z f(z)\,dz$$

where $\int_a^z$ means $\int_\gamma$ for any contour γ in A from a to z.

The vanishing integral hypothesis shows that F is well defined, and as in the proof of the Primitive Theorem 2.5, $F' = f$. In particular F is analytic, so by Cauchy's derivative theorem, so is $f = F'$. $\square$

2.14 Corollary *If f is continuous on a region A and analytic on $A - \{a\}$ for some point $a \in A$, then f is in fact analytic at a as well.*

Proof This is immediate from the generalized Cauchy Theorem and Morera's Theorem applied to f on an open disk in A containing a. $\square$

2.15 Liouville's Theorem *If $f : \mathbb{C} \to \mathbb{C}$ is analytic and bounded (meaning there exists a constant M with $|f(z)| < M$ for all $z \in \mathbb{C}$) then f is constant.*

Proof For any $z \in \mathbb{C}$ and any $r > 0$, we have

$$|f'(z)| \;=\; \left| \frac{1}{2\pi i} \int_{\partial D_r(z)} \frac{f(\zeta)}{(\zeta - z)^2}\, d\zeta \right| \;\leqslant\; \frac{1}{2\pi}\, 2\pi r\, \frac{M}{r^2} \;=\; \frac{M}{r}$$

by Cauchy's Derivative Theorem and the Estimation Theorem. Since this goes to 0 as $r \to \infty$, this shows that $f'(z) = 0$ for all z, and so f is constant. □

Remarks ⓐ The proof shows that if f is analytic on a disk D of radius r, and assumes a maximum value of M on ∂D, then $|f'(z)| < M/r$. A similar argument using the higher order derivative formulas shows that $|f^{(k)}(z)| \leqslant k!M/r^k$ for all $k \geqslant 0$. These are known as Cauchy's Inequalities.

ⓑ An analytic function defined on all of $\mathbb{C}$ is also called an entire function. Thus Liouville's Theorem says that *bounded entire functions are constant.*

2.16 Fundamental Theorem of Algebra *Any non-constant complex polynomial*

$$p(z) \;=\; p_0 + p_1 x + \cdots + p_n z^n$$

has at least one complex root.

Remark If z_1 is such a root, then $p(z)$ factors as $(z - z_1)q(z)$ for some polynomial q of degree $n - 1$. Repeating, we see that there is a factorization $p(z) = p_n(z - z_1)\cdots(z - z_n)$, where $z_1, \ldots, z_n$ are the roots of p "with multiplicities" (i.e. each root z_i is repeated m_i times, where m_i, its multiplicity, is the least positive integer such that $p^{(m_i)}(z_i) \neq 0$).

Proof If p were never zero, then $f = 1/p$ would be an entire function. Furthermore, f would be bounded. Indeed

$$\lim_{z\to\infty} f(z) \;=\; \lim_{z\to\infty} \frac{1}{p_0 + \cdots + p_n z^n} \;=\; \lim_{z\to\infty} \frac{1/z^n}{p_0/z^n + \cdots + p_n} \;=\; 0/p_n \;=\; 0$$

and so f would be bounded outside some closed disk $D \subset \mathbb{C}$, but f is certainly bounded on D since it is continuous and D is compact. This would force f to be constant, by Liouville's Theorem, and so p would be constant, a contradiction. □

***2.17 Lucas's Theorem** *The smallest convex polygon that contains all the roots of a polynomial $p(z)$ also contains all the roots of its derivative $p'(z)$.* (This is the analogue of Rolle's Theorem in real variable calculus.)

Proof It suffices to show that if all the roots of p lie in a closed half plane H, and $z \notin H$, then $p'(z) \neq 0$. Rotating H by a suitable angle θ converts it into an *upper* half-plane at some height r above the real axis, and so $w \in H \iff \mathrm{Im}(\omega z) \geqslant r$ where $\omega = \exp(i\theta)$. In particular, for any root z_i of p we have

$$\mathrm{Im}(\omega(z - z_i)) \;=\; \mathrm{Im}(\omega z) - \mathrm{Im}(\omega z_i) \;<\; 0$$

and so $\mathrm{Im}(1/\omega(z - z_i)) > 0$. A quick calculation shows that if $z_1, \ldots, z_n$ is the full list of roots of p with multiplicities, i.e. $p(z) = p_n(z - z_1)\cdots(z - z_n)$, then

$$\frac{p'(z)}{p(z)} \;=\; \frac{1}{z - z_1} + \cdots + \frac{1}{z - z_n}.$$

It follows that $\mathrm{Im}(p'(z)/\omega p(z)) > 0$, and so $p'(z) \neq 0$. □

Computing integrals of rational functions

Let $f(z) = p(z)/q(z)$ be a rational function, that is $p(z)$ and $q(z)$ are polynomials, which we can take to be relatively prime. We wish to compute the integral $\int_\gamma f$ for any simple closed contour γ on which f is defined (i.e. on which q is nonzero). To do so in general, we must assume that we know the roots $z_1, \ldots, z_n$ of q, and their multiplicities $m_1, \ldots, m_n$. This means that for each $i = 1, \ldots, n$, we can write $f(z) = f_i(z)/(z - z_i)^{m_i}$, where $f_i(z) = f(z)(z - z_i)^{m_i}$ is a rational function that does not have z_i as a root of its denominator.

Now suppose that the roots $z_1, \ldots, z_k$ lie inside γ, while $z_{k+1}, \ldots, z_n$ lie outside. Choose small disks D_i centered at the z_i, and arcs α_i joining ∂D_i to γ. We can assume that these disks and arcs are disjoint (except at the starting points of the arcs) and that they all lie inside γ (except at the endpoints of the arcs). This partitions γ into subarcs $\gamma_1, \ldots, \gamma_k$ so that $\gamma = \gamma_1 + \cdots \gamma_k$, and produces a null-homotopic curve $\tau_1 + \cdots + \tau_k$ in $\mathbb{C} - \{z_1, \ldots, z_n\}$, where $\tau_i = \gamma_i - \alpha_i - \partial D_i + \alpha_i$, as shown in the figure. We say γ is homologous to $\sum \partial D_i$

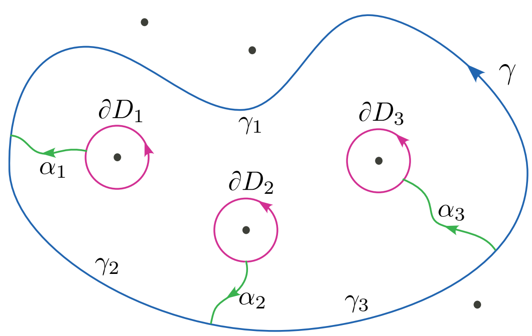

Figure 8. Homologous contours

(written $\gamma \sim \sum \partial D_i$) in $\mathbb{C} - \{z_1, \ldots, z_n\}$. By Cauchy's Theorem and Derivative Formula,

$$\int_\gamma f(z)\,dz \;=\; \sum_{i=1}^{k} \int_{\partial D_i} f(z)\,dz \;=\; \sum_{i=1}^{k} \int_{\partial D_i} \frac{f_i(z)}{(z - z_i)^{m_i}} \;=\; \sum_{i=1}^{k} \frac{2\pi i}{(m_i - 1)!}\, f_i^{(m_i-1)}(z_i).$$

E. Proof of Cauchy's Theorem

We begin with a very special case, proved by Goursat in 1883.

2.18 **Goursat's Lemma** (Cauchy's Theorem for a Triangle) *If f is analytic in a region A and $\triangle$ is a triangle in A, then $\int_{\partial\triangle} f = 0$.*

Proof Construct a sequence of triangles $\triangle = \triangle_0 \supset \triangle_1 \supset \triangle_2 \supset \cdots$ [†] with boundary lengths $L = L_0 > L_1 > L_2 > \cdots$ such that

$$L_n \;=\; \frac{1}{2^n}\, L \qquad \text{and} \qquad \left| \int_{\partial\triangle_n} f \right| \;\geqslant\; \frac{1}{4^n} \left| \int_{\partial\triangle} f \right|.$$

[†] Divide $\triangle$ into four congruent triangles $\triangle^k$ ($k = 1, \ldots, 4$) oriented so that $\int_\triangle f = \sum_k \int_{\partial\triangle^k} f$. Then $|\int_\triangle f| \leqslant \sum_k |\int_{\partial\triangle^k} f|$. Let $\triangle_1 = \triangle^k$ with $|\int_{\triangle^k} f|$ maximal. Continue with $\triangle_1$ in place of $\triangle$, etc.

Choose a point $a \in A$ in the intersection of all the $\triangle_n$'s (it exists because the $\triangle_n$'s are compact). Since f is differentiable at a, every $\varepsilon > 0$ has a $\delta > 0$ such that

$$|z - a| < \delta \implies |f(z) - f(a) - f'(a)(z - a)| \ < \ \varepsilon|z - a|.$$

For n satisfying $L/2^n < \delta$, we have $|z - a| < L_n = L/2^n < \delta$ for every $z \in \partial\triangle_n$, and so

$$\left| \int_{\partial\triangle_n} f \right| \ = \ \left| \int_{\partial\triangle_n} (f(z) - f(a) - f'(a)(z - a))\, dz \right| \ \leqslant \ \varepsilon L_n^2 \ = \ \varepsilon L^2/4^n$$

(The first equality follows from Theorem 2.1, since $f(a)$ and $f'(a)(z - a)$ have primitives and $\partial\triangle_n$ is closed, and the inequality follows from the Estimation Theorem.) It follows that $|\int_{\partial\triangle} f| \leqslant \varepsilon L^2$. Since this is true for all ε, we have $\int_{\partial\triangle} f = 0$. □

All the other versions of Cauchy's Theorem follow from Goursat's Lemma.

Proof of the generalized Goursat Lemma for a Triangle Let a be a "bad" point inside $\triangle$ where we only assume that f is continuous. For any $\varepsilon > 0$, consider a small triangle $\triangle'$ around a with boundary length less than ε/M, where M is the maximum modulus of f on $\triangle$. Now chop up the rest of $\triangle$ into triangles $\triangle_1, \triangle_2, \ldots$, so $\int_{\partial\triangle_k} f = 0$ for all k by Goursat's Lemma. Hence

$$\left| \int_{\triangle} f \right| \ = \ \left| \int_{\triangle'} f + \sum_k \int_{\triangle_k} f \right| \ = \ \left| \int_{\triangle'} f \right| \ \leqslant \ \frac{\varepsilon}{M} M \ = \ \varepsilon.$$

Since ε was arbitrary, $\int_{\triangle} f = 0$. □

Proof of Cauchy's Theorem in a Disk D (This includes the generalized version.) Pick a point a in D, and for every $z \in D$ let $[a, z]$ be the oriented line segment from a to z. Set

$$F(z) \ = \ \int_{[a,z]} f.$$

By Goursat's Lemma $F(z + h) - F(z) = \int_{[z,z+h]} f$ for any small h, and so as in the proof of the Primitive Theorem 2.5, we see that F is an antiderivative of f in D. Therefore $\int_\gamma f = 0$ for all closed γ in D, by the Fundamental Theorem of Contour Integrals. □

Proof of the Deformation Theorem (Sketch) Let

$$H : R \longrightarrow A \qquad \text{where} \qquad R \ = \ [0, 1] \times [0, 1] \quad (\text{“}R\text{” for “rectangle”})$$

be a homotopy between contours $\gamma_0, \gamma_1 : [0, 1] \to A$, keeping the endpoints $z = \gamma_0(0) = \gamma_1(0)$ and $w = \gamma_i(1) = \gamma_1(1)$ fixed. There exist open disks sets $D_k \subset A$, for $k = 1, \ldots, m$, such that $H(R) \subset D_1 \cup \cdots \cup D_m$ (note that $H(R)$ is compact since R is compact), and so $R = H^{-1}(D_1) \cup \cdots \cup H^{-1}(D_m)$.

Pick n so large that each rectangle $R_{p,q} = [p/n, (p+1)/n] \times [q/n, (q+1)/n] \subset H^{-1}(D_k)$, or equivalently $H(R_{p,q}) \subset D_k$, for some k. Set $\gamma = H|\partial R$ and $\gamma_{p,q} = H|\partial R_{p,q}$. It can be arranged (adjusting H and n) that γ and $\gamma_{p,q}$ are piecewise smooth.

It follows from Cauchy's Theorem in a disk that

$$\int_{\gamma_{p,q}} f = 0 \qquad \text{and so} \qquad \int_\gamma f \ = \ \sum_{p,q} \left(\int_{\gamma_{p,q}} f \right) \ = \ 0.$$

But setting $\lambda_i(t) = H(i, t)$ for $i = 0$ and 1 (which are constant paths at z an w) we have $\gamma = \gamma_0 + \lambda_1 - \gamma_1 - \lambda_0$ and so $0 = \int_\gamma f = \int_{\gamma_0} f + 0 - \int_{\gamma_1} f - 0$. Therefore $\int_{\gamma_0} f \ = \ \int_{\gamma_1} f$. □

3. Series

A. Basic Notions

Numerical Sequences and Series

A sequence $z_0, z_1, z_2, \ldots$ (abbreviated z_n) of complex numbers is said to converge to $z \in \mathbb{C}$, written $\lim_{n\to\infty} z_n = z$, or $z_n \to z$, if

$$\forall\, \varepsilon > 0,\ \exists\, N \text{ such that } n > N \Longrightarrow |z_n - z| < \varepsilon.$$

If $\forall\, M,\ \exists\, N$ such that $n > N \Longrightarrow |z_n| > M$, we say z_n diverges to ∞, and write $\lim_{n\to\infty} z_n = \infty$, or $z_n \to \infty$. In either case, the limit z is unique if it exists ($\triangle$ inequality).

A priori weaker than convergence is the condition that z_n be a Cauchy sequence, i.e.

$$\forall\, \varepsilon > 0,\ \exists\, N \text{ such that } p, q > N \Longrightarrow |z_p - z_q| < \varepsilon,$$

also written $\lim_{p,q\to\infty}(z_p - z_q) = 0$, or simply $z_p - z_q \to 0$. But the completeness of $\mathbb{R}$ shows that these notions are in fact equivalent: every Cauchy sequence converges to a unique complex number, and conversely, every convergent sequence is Cauchy – an easy exercise again using the $\triangle$ inequality. So we accept without proof this powerful criterion for convergence, useful because it does not require a knowledge of the limiting value.

3.1 a) Cauchy Criterion *z_n converges $\Longleftrightarrow$ z_n is a Cauchy sequence.*

Remark The usual limit theorems hold: The limit of a sum of two convergent sequences is the sum of the limits, and similarly for differences, products and quotients (when the denominator's limit is nonzero). Also if $z_n \to z$, then $\frac{1}{n}(z_1 + \cdots z_n) \to z$ as well (HW).

A series $z_1 + z_2 + z_3 + \cdots$ (abbreviated $\sum_{n=0}^{\infty} z_n$) converges to z, written $\sum_{n=0}^{\infty} z_n = z$, if its sequence of partial sums $z_1,\ z_1 + z_2,\ z_1 + z_2 + z_3, \ldots$ converges to z. The Cauchy criterion translates into the following:

3.1 b) Cauchy Criterion for Series *$\sum_{n=0}^{\infty} z_n$ converges $\Longleftrightarrow \forall\, \varepsilon > 0,\ \exists\, N$ such that $N < p \leqslant q \Longrightarrow |\sum_{n=p}^{q} z_n| < \varepsilon$, which we also simply write as $\sum_{n=p}^{q} z_n \to 0$.*

A series $\sum_{n=0}^{\infty} z_n$ converges absolutely if the associated real series $\sum_{n=0}^{\infty} |z_n|$ converges. Since

$$|\sum_{n=p}^{q} z_n| \ \leqslant\ \sum_{n=p}^{q} |z_n|\,,$$

the Cauchy criterion for series shows that ***absolute convergence implies convergence***. Thus all the tests (ratio, root, integral, comparison) for convergence of real series can be used in analyzing complex series.

Example Fix $z \in \mathbb{C}$. Then $\sum_{n=0}^{\infty} z^n/n!$ converges absolutely, by the ratio test:

$$\frac{|z^{n+1}/(n+1)!|}{|z^n/n!|} \ =\ \frac{|z|}{n+1} \ \longrightarrow\ 0.$$

Below we show that this series converges to e^z.

Sequences and Series of Functions

A sequence of complex functions $f_n : A \to \mathbb{C}$ is said to converge pointwise to f, written $\lim_{n\to\infty} f_n = f$ or $f_n \to f$, if $f_n(z) \to f(z)$ for every $z \in A$. It converges uniformly to f, written $\lim_{n\to\infty} f_n =_u f$ or $f_n \to_u f$, if $\forall\, \varepsilon > 0,\ \exists\, N$ such that $n > N \Longrightarrow |f_n(z) - f(z)| < \varepsilon$

for all $z \in A$. The difference is that for uniform convergence, N is independent of z. If $f_n \to_u f$ on all compact subsets of A (or equivalently on all closed disks in A), then we say f_n converges almost uniformly to f on A, and write $f_n \to_{au} f$.

Example The sequence $f_n(z) = |z|^n$ converges *pointwise* on the closed unit disk D to the function that is 1 on the boundary ∂D and 0 on the inside D°. This convergence is *not uniform*, since if it were, then for any $\varepsilon > 0$ there would exist an n such that $|z|^n < \varepsilon$ for all $z \in D^\circ$. But for $z = \varepsilon^{1/n}$ this would imply $\varepsilon < \varepsilon$, a contradiction. This example illustrates the fact that the limit of a sequence of continuous functions need not be continuous. We will see below, however, that continuity is preserved under uniform limits.

The series $\sum_{n=0}^{\infty} f_n$ is said to converge pointwise to f, written $\sum_{n=0}^{\infty} f_n = f$, if the sequence of partial sums $f_0,\ f_0 + f_1,\ f_0 + f_1 + f_2, \ldots$ converges pointwise to f. Similarly for uniform (or almost uniform) convergence, written $\sum_{n=0}^{\infty} f_n =_u f$ (or $\sum_{n=0}^{\infty} f_n =_{au} f$).

For pointwise convergence, the Cauchy criterion can be applied at each point in A separately. For uniform convergence , we have the following:

3.2 Uniform Cauchy Criterion a) (for sequences) *f_n converges uniformly* $\Longleftrightarrow$ $\forall\, \varepsilon > 0,\ \exists\, N$ *such that* $p, q > N \Longrightarrow |f_p(z) - f_q(z)| < \varepsilon$ *for all* $z \in A$.

b) (for series) $\sum_{n=0}^{\infty} f_n$ *converges uniformly* $\Longleftrightarrow \forall\varepsilon > 0, \exists\, N$ *such that* $N < p \leqslant q \Longrightarrow$ $|\sum_{n=p}^{q} f_n(z)| < \varepsilon$ *for all* $z \in A$.

Proof **b)** follows from **a)** applied to the partial sums. For **a)** ($\Longrightarrow$) suppose $f_n \to_u f$. Fix $\varepsilon > 0$. Then $\exists\, N$ such that $n > N \Longrightarrow |f_n(z) - f(z)| < \varepsilon/2$ for all $z \in A$, and so

$$|f_p(z) - f_q(z)| \ \leqslant\ |f_p(z) - f(z)| + |f(z) - f_q(z)| \ <\ \varepsilon/2 + \varepsilon/2 \ =\ \varepsilon$$

for all $p, q > N$ and $z \in A$, by the $\triangle$ inequality.

($\Longleftarrow$) Let f be the pointwise limit of f_n, which exists by the Cauchy Criterion 3.1a. Fix $\varepsilon > 0$. Choose N so that $p, q > N \Longrightarrow |f_p(z) - f_q(z)| < \varepsilon/2$ for all $z \in A$. Now fix $z \in A$, and choose $q > N$ so that $|f_q(z) - f(z)| < \varepsilon/2$. Then $p > N \Longrightarrow$

$$|f_p(z) - f(z)| \ \leqslant\ |f_p(z) - f_q(z)| + |f_q(z) - f(z)| \ <\ \varepsilon/2 + \varepsilon/2 \ =\ \varepsilon.$$

Since N is independent of z, we have $f_p \to_u f$. □

3.3 Weierstrass *M*-test (Uniform comparison test) *Let $f_n : A \to \mathbb{C}$ be a sequence of functions. If $\exists\, M_n \in \mathbb{R}$ such that $\sum M_n$ converges and $|f_n(z)| < M_n$ for all $z \in A$, then $\sum f_n$ converses absolutely and uniformly on A.*

Proof Let $\varepsilon > 0$. Choose N such that $q \geqslant p > N \Longrightarrow \sum_{n=p}^{q} M_n < \varepsilon$ (the Cauchy criterion for the convergence of $\sum M_n$). Then

$$|\sum_{n=p}^{q} f_n(z)| \ \leqslant\ \sum_{n=p}^{q} |f_n(z)| \ <\ \sum_{n=p}^{q} M_n \ <\ \varepsilon\,.$$

The result follows by 3.2b. □

Example Fix $r < 1$. Then the M-test with $M_n = r^n$ shows that the series $\sum_{n=0}^{\infty} z^n$ converges uniformly to $1/(1-z)$ for $|z| < r$, and so almost uniformly for $|z| < 1$.

3.4 Weierstrass' Theorem *Let $f_n : A \to \mathbb{C}$ be a sequence of functions.*

a) *If f_n converges uniformly to a function f, then*

- *f_n continuous $\Longrightarrow f$ continuous and $\int_\gamma f_n \to \int_\gamma f$ for any contour γ in A*
- *f_n analytic $\Longrightarrow f$ analytic and $f_n' \to_{au} f'$* †

b) *If $\sum f_n$ converges uniformly to a function f, then*

- *f_n continuous $\Longrightarrow f$ continuous and $\sum \int_\gamma f_n = \int_\gamma f$ for any contour γ in A*
- *f_n analytic $\Longrightarrow f$ analytic and $\sum f_n' =_{au} f'$.*

Thus one can integrate or differentiate a uniformly convergent series term by term.

Proof **a)** First assume the f_n are continuous, and fix $\varepsilon > 0$. Since $f_n \to_u f$, by hypothesis, we can choose n so that $|f_n(z) - f(z)| < \varepsilon/3$ for all $z \in A$. Now for any $a \in A$, the continuity of f_n at $a \Longrightarrow \exists\, \delta > 0$ such that $|f_n(z) - f_n(a)| < \varepsilon/3$ whenever $|z - a| < \delta$, so in that case

$$\begin{aligned} |f(z) - f(a)| &\leqslant |f(z) - f_n(z)| + |f_n(z) - f_n(a)| + |f_n(a) - f(a)| \\ &< \varepsilon/3 + \varepsilon/3 + \varepsilon/3 \;=\; \varepsilon \end{aligned}$$

Therefore f is continuous at a.

Next choose N so that $n > N \Longrightarrow |f_n(z) - f(z)| < \varepsilon/L$ for all $z \in A$, where L is the length of γ. Then $\left|\int_\gamma f_n - \int_\gamma f\right| = \left|\int_\gamma (f_n - f)\right| < L(\varepsilon/L) = \varepsilon$ and so $\int_\gamma f_n \to \int_\gamma f$.

Next assume the f_n are analytic. Then certainly the f_n are continuous, and so f is continuous by the previous argument. By Cauchy's Theorem, $\int_\gamma f_n = 0$ for every null-homotopic closed curve γ in A. This implies $\int_\gamma f = 0$ by the previous argument, and so f is analytic, by Morera's Theorem.

Now fix closed disks $D \subset A$ of radius r and $E \subset \operatorname{int} D$, and $\varepsilon > 0$. Then for any $z \in E$, we have

$$\begin{aligned} |f_n'(z) - f'(z)| &= \left| \frac{1}{2\pi i} \int_{\partial D} \frac{f_n(\zeta) - f(\zeta)}{(\zeta - z)^2} d\zeta \right| \qquad \text{(by CIF)} \\ &< \frac{1}{2\pi} 2\pi r \frac{\varepsilon r}{r^2} \;=\; \varepsilon \end{aligned}$$

for n chosen large enough so that $|f_n(\zeta) - f(\zeta)| < \varepsilon r$ for $\zeta \in \partial D$ (uniform convergence of $f_n \to f$ on ∂D). Thus $f_n' \to f'$ uniformly on E, and thus almost uniformly on A.

For **b)**, apply **a)** to the sequence of partial sums (exercise). □

Example The Riemann ζ-function

$$\zeta(z) \;=\; \sum_{n=1}^{\infty} \frac{1}{n^z}$$

is analytic in the open right half plane $H = \{z : \operatorname{Re}(z) > 1\}$.

Proof It suffices to show that $\zeta(z)$ converges uniformly on any closed disk $D \subset H$. Clearly $\exists\, p \in \mathbb{R}$ with $p > 1$ such that $\operatorname{Re}(z) > p$ for all $z \in D$. Also

$$|1/n^z| \;=\; 1/n^{\operatorname{Re}(z)} \;<\; 1/n^p$$

Now taking $M_n = 1/n^p$, the Weierstrass M-test gives uniform convergence. □

† In short: $\lim(\int f_n) = \int(\lim f_n)$ and $\lim f_n' = (\lim f_n)'$

B. Taylor's and Laurent's Theorems

Let $f : A \to \mathbb{C}$ be analytic.

3.5 Taylor's Theorem *If D is an open disk*[†] *in A with center a, then for all $z \in D$,*

$$f(z) \underset{au}{=} \sum_{n=0}^{\infty} f_n(z-a)^n \qquad \textit{where} \quad f_n = \frac{1}{2\pi i}\int_\gamma \frac{f(\zeta)}{(\zeta-a)^{n+1}}\,d\zeta$$

for any (positively oriented) circle γ in D with center a.

Remark Cauchy's Derivative Formula shows that f_n can be expressed in terms of the nth derivative of f at a: $f_n = f^{(n)}(a)/n!$. This yields the familiar expansion

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(z-a)^n \qquad \text{for all } z \in D.$$

As an immediate consequence we see that if f and all its derivatives vanish at a, then $f \equiv 0$ on D. On the other hand, if $f^{(k)}(a) \neq 0$ for some (smallest) k, then we claim that $f(z) \neq 0$ for all $z \neq a$ sufficiently close to a; indeed Taylor's theorem gives $f(z) = (z-a)^k\varphi(z)$, where $\varphi : D \to \mathbb{C}$ is analytic with $\varphi(a) \neq 0$, and the claim follows from the continuity of φ. This implies the remarkable "rigidity" property of analytic functions, that they are determined by their values on "small" subsets of their domains:

Identity Theorem *If two analytic functions g and h defined on an open connected set A agree on a sequence of points in A that converge to a point in A, then $g = h$ everywhere in A. In particular the zeros of any nonconstant analytic function $f : A \to \mathbb{C}$ are isolated.*

Proof The first statement follows from the second applied to $f = g - h$, so we prove the latter. As noted above, if a is a zero of f that is not isolated, then $f \equiv 0$ in some neighborhood of a. Therefore the set of non-isolated zeros form an open (and clearly closed) subset of A, and so must be empty by the connectedness of A, since f is nonconstant. $\square$

3.6 Laurent's Theorem *If R is an open annulus*[†] *in A with center a (i.e. the region between two concentric circles centered at a) then for all $z \in R$,*

$$f(z) \underset{au}{=} \sum_{n=-\infty}^{\infty} f_n(z-a)^n$$

for f_n as defined in Taylor's Theorem, where γ is any circle in A centered at a.

To prove Taylor's and Laurent's theorems, we appeal to the following:

3.7 Lemma *If z is any point not on γ, then*

$$\frac{1}{2\pi i}\int_\gamma \frac{f(\zeta)}{\zeta - z}\,d\zeta = \begin{cases} \sum_{n\geqslant 0} f_n(z-a)^n & \textit{for } z \textit{ inside } \gamma \\ \sum_{n<0} f_n(z-a)^n & \textit{for } z \textit{ outside } \gamma \end{cases}$$

Note: we only need continuity of f on γ for the lemma.

[†] The disk D in 3.5 can be replaced by $\mathbb{C}$, and the annulus R in 3.6 can be replaced by the region outside a circle, a punctured disk, or $\mathbb{C}-$point. In other words, the radii can go to 0 or to ∞.

Proof For z inside γ we compute

$$\frac{1}{\zeta - z} \;=\; \frac{1}{\zeta - a}\,\frac{1}{1-(z-a)/(\zeta-a)} \;\underset{au}{=}\; \sum_{n\geqslant 0} \frac{(z-a)^n}{(\zeta-a)^{n+1}}$$

by the Weierstrass M-test (see the example below 3.3), and so by Theorem 3.4b

$$\frac{1}{2\pi i}\int_\gamma \frac{f(\zeta)}{\zeta - z}\,d\zeta \;=\; \sum_{n\geqslant 0}\left(\frac{1}{2\pi i}\int_\gamma \frac{f(\zeta)}{(\zeta-a)^{n+1}}\,d\zeta\right)(z-a)^n \;=\; \sum_{n\geqslant 0} f_n(z-a)^n\,.$$

Interchanging the role of ζ and z, we see that for z outside γ we have

$$\frac{1}{\zeta - z} \;=\; -\frac{1}{z-\zeta} \;\underset{au}{=}\; -\sum_{k\geqslant 0}\frac{(\zeta-a)^k}{(z-a)^{k+1}} \;=\; -\sum_{n<0}\frac{(z-a)^n}{(\zeta-a)^{n+1}}\,.$$

where the last equality follows by setting $n = -(k+1)$. It follows exactly as above that

$$\frac{1}{2\pi i}\int_\gamma \frac{f(\zeta)}{\zeta - z}\,d\zeta \;\underset{au}{=}\; \sum_{n<0} f_n(z-a)^n$$

for z outside γ. □

Proof (of Taylor's Theorem) Fix $z \in D$. Let γ be a circle in D with center at a and z inside γ. Then by Lemma 3.7,

$$f(z) \;\underset{\text{CIF}}{=}\; \frac{1}{2\pi i}\int_\gamma \frac{f(\zeta)}{\zeta - z}\,d\zeta \;\underset{au}{=}\; \sum_{n=0}^{\infty} f_n(z-a)^n\,. \quad \square$$

Proof (of Laurent's Theorem) Fix $z \in R$. Let γ and Γ be concentric circles in D centered at a with z outside γ but inside Γ. Then by Lemma 3.7,

$$f(z) \;\underset{\text{CIF}}{=}\; \frac{1}{2\pi i}\int_\Gamma \frac{f(\zeta)}{\zeta - z}\,d\zeta - \frac{1}{2\pi i}\int_\gamma \frac{f(\zeta)}{\zeta - z}\,d\zeta \;\underset{au}{=}\; \sum_{n=-\infty}^{\infty} f_n(z-a)^n\,. \quad \square$$

Example Since exp is entire and $\exp^{(n)}(0) = \exp(0) = 1$, we have

$$e^z \;=\; \textstyle\sum_{n\geqslant 0} z^n/n!$$

for all $z \in \mathbb{C}$; the convergence is uniform on any compact subset of $\mathbb{C}$. This is the Taylor expansion of e^z about the origin. We can use it to find the Laurent expansion of $e^{1/z}$ in $\mathbb{C} - \{0\}$, as follows: For all $z \neq 0$ in $\mathbb{C}$,

$$e^{1/z} \;=\; \textstyle\sum_{n\geqslant 0} (1/z)^n/n! \;=\; \sum_{n\leqslant 0} z^n/|n|!\,.$$

Note that it is difficult to find this directly, since the integrals defining the coefficients are difficult to compute. However, if we knew the uniqueness of the Laurent expansion (see below) then we could conclude that $\oint z^{n-1}e^{1/z}\,dz \;=\; 2\pi i/n!$ for all $n \geqslant 0$.

C. Power Series and Laurent Series

Definition A Laurent series is an infinite series of the form

$$\sum_{n=-\infty}^{\infty} a_n(z-a)^n.$$

We say that the series is centered at a. If $a_n = 0$ for all $n < 0$, then this is simply called a power series. If $a_n = 0$ for all $n \geqslant 0$ it is called a negative power series.

Power Series

The radius of convergence of a power series $\sum_{n\geqslant 0} a_n(z-a)^n$ is

$$R \;:=\; \sup\{\, r \geqslant 0 \;:\; \sum_{n\geqslant 0} |a_n|\, r^n \text{ converges} \,\}^{\dagger}$$

which is a non-negative real number or ∞. The open disk $D_R(a)$ is called the disk of convergence of the series, and the set of all $z \in \mathbb{C}$ at which the series converges is called its domain of convergence.

3.8 Theorem *The power series $\sum_{n\geqslant 0} a_n(z-a)^n$ converges absolutely and almost uniformly on its disk D of convergence, and diverges on $\mathbb{C} - \overline{D}$.*

It follows that the domain C of convergence of the series satisfies $D \subset C \subset \overline{D}$.

Proof Let $D = D_R(a)$, and fix $r < R$. By definition of R, we can find $w \in D$ at a distance $s > r$ from a such that $\sum a_n(w-a)^n$ converges. Then $a_n(w-z)^n \to 0$, and so $\exists\, M$ such that $|a_n(w-a)^n| < M$ for all n. Now if $|z-a| \leqslant r$, then

$$|a_n(z-a)^n| \;=\; |a_n(w-a)^n| \left|\frac{z-a}{w-a}\right|^n \;<\; M\left(\frac{r}{s}\right)^n .$$

Since $\sum M(r/s)^n$ is a convergent (geometric) series, the M-test shows that $\sum a_n(z-a)^n$ converges uniformly and absolutely on $\overline{D}_r(a)$.

For the last assertion, note that if the series converges at a point w at a distance $s > R$ from a, then the argument above shows that it also converges absolutely at any closer point to a, contradicting the definition of R. $\square$

3.9 Corollary a) *The power series $\sum_{n\geqslant 0} a_n(z-a)^n$ defines an analytic function $f(z)$ within in its disk D of convergence, with $f'(z) = \sum_{n\geqslant 0} na_n(z-a)^{n-1}$. These two series have the same radius of convergence. Furthermore,*

$$a_k \;=\; \frac{1}{2\pi i}\int_\gamma \frac{f(\zeta)}{(\zeta-a)^{k+1}}\,d\zeta \;=\; f^{(k)}(a)/k!$$

for each $k \geqslant 0$.

b) *A complex function is analytic at a point $a \in \mathbb{C} \iff$ it can be expanded in a power series $\sum a_n(z-a)^n$ in some open disk centered at a. The series expansion in $(\Longrightarrow)$ is unique, and valid in any open disk lying in the domain of analyticity of the function.*

Proof **a)** The first statement follows from the theorem and Weierstrass' Theorem 3.4b. To prove the second, suppose that the derived series converged at some w outside $\overline{D}$. Then $|na_n(w-a)^{n-1}| \to 0 \Longrightarrow |a_n(w-a)^n| \to 0$, which would imply as in the proof of 3.8 that the original series converged absolutely at any point closer to a than w, and therefore at points not in $\overline{D}$, a contradiction. The last statement follows from Taylor's Theorem and the Cauchy Derivative Formula.

b) $(\Longrightarrow)$ follows from Taylor's Theorem, and $(\Longleftarrow)$ from part a) of this corollary. The last statement also follows from these two results. $\square$

† The classical Cauchy-Hadamard Theorem gives the explicit formula $R = (\limsup |a_n|^{1/n})^{-1}$.

Negative Power Series

The radius of divergence of a negative power series $\sum_{n<0} a_n(z-a)^n$ is

$$S \ := \ \inf\{\, s \geqslant 0 \ : \ \sum_{n<0} |a_n|\, s^n \text{ converges}\,\}.$$

Thus $S \geqslant 0$ when the set on the right is nonempty, or ∞ (by definition) when it is empty. The (infinite) open annulus $A = \{z \,:\, |z| > S\}$ is called the annulus of convergence of the series, and its complement, the open disk $D_S(a)$, is called its disk of divergence. The set of all $z \in \mathbb{C}$ at which the series converges is called its domain of convergence.

3.8′ Theorem *The negative series $\sum_{n<0} a_n(z-a)^n$ converges absolutely and almost uniformly on its annulus A of convergence, and diverges on $\mathbb{C} - \overline{A}$.*

It follows that the domain C of convergence of the series satisfies $A \subset C \subset \overline{A}$.

3.9′ Corollary *The negative power series $\sum_{n<0} a_n(z-a)^n$ defines an analytic function $f(z)$ within in its annulus A of convergence, with $f'(z) = \sum_{n<0} na_n(z-a)^{n-1}$. These two series have the same radius of divergence. Furthermore, for any positively oriented circle γ in A centered at a we have*

$$a_k \ = \ \frac{1}{2\pi i}\int_\gamma \frac{f(\zeta)}{(\zeta-a)^{k+1}}\, d\zeta$$

for each $k < 0$, and so this negative series expansion of f is unique.

The proofs of 3.8′ and 3.9′ are analogous to 3.8 and 3.9 (exercise).

Remark Perhaps it is clearer to replace a_n by b_{-n}, and so

$$f(z) \ = \ \sum_{n\geqslant 1}\frac{b_n}{(z-a)^n} \ = \ \frac{b_1}{z-a} + \frac{b_2}{(z-a)^2} + \cdots$$

$$f'(z) \ = \ -\frac{b_1}{(z-a)^2} - \frac{2b_2}{(z-a)^3} - \cdots$$

where $2\pi i b_n = \int_\gamma f(\zeta)(\zeta-a)^{n-1}\, d\zeta$ for each $n \geqslant 1$.

Laurent Series

A Laurent series $\sum_{n=-\infty}^{\infty} a_n(z-a)^n$ is just the sum of a positive power series – say with radius of convergence R – and a negative one – say with radius of divergence S. The negative part is called the principal part of the Laurent series.

If $R > S$, then the Laurent series converges in the annulus $\{z \,:\, S < |z-a| < R\}$ and is unique by Corollaries 3.9 and 3.9′. From Laurent's Theorem 3.6, we conclude:

3.10 Theorem *For any complex analytic function $f : A \to \mathbb{C}$ and any point $a \in \mathbb{C}$ (not necessarily in A), the function f can be expanded uniquely in a Laurent series within any open annulus contained in A and centered at a.*

Computing Taylor and Laurent series

It is generally inconvenient (if not impossible) to compute these series directly from the formulas for their coefficients given in Theorems 3.8 and 3.8′. There are several other methods for making these computations. We illustrate these using the known Taylor series 1 $e^z = \sum_{n\geqslant 0} z^n/n!$ on $\mathbb{C}$, and 2 $1/(1-z) = \sum_{n\geqslant 0} z^n$ on $|z| < 1$.

Substitute in a known series (a) (from ①) $e^{1/z} = 1 + \frac{1}{z} + \frac{1}{2!z^2} + \cdots$ for $z \neq 0$.

(b) (from ②) $\frac{1}{1+z} = 1 - z + z^2 - + \cdots$ for $|z| < 1$.

Multiply series together (or add, subtract, or divide)

$$\sum_{n \geqslant 0} a_n (z-a)^n \cdot \sum_{n \geqslant 0} b_n (z-a)^n = \sum_{n \geqslant 0} \left(\sum_{k=0}^{n} a_k b_{n-k} \right) (z-a)^n$$

with radius of convergence $\geqslant$ the minimum of the two radii for the series on the left.

(a) (from ①) $ze^z = z \sum_{n \geqslant 0} \frac{z^n}{n!} = z + z^2 + \frac{z^3}{2!} + \frac{z^4}{3!} + \cdots$

(b) (from ①) Since $e^{iz} = \cos z + i \sin z = 1 + iz - \frac{z^2}{2!} - i\frac{z^3}{3!} + \frac{z^4}{4!} + i\frac{z^5}{5!} - - + + \cdots$,

$$\cos z = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - + \cdots \qquad \text{and} \qquad \sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - + \cdots$$

and so

$$\begin{aligned} \cos z \sin z &= z - \left(\frac{1}{2!} + \frac{1}{3!} \right) z^3 + \left(\frac{1}{4!} + \frac{1}{2!3!} + \frac{1}{5!} \right) z^5 - + \cdots \\ &= \frac{1}{2} \sin 2z = \frac{1}{2} \left(2z - \frac{(2z)^3}{3!} + - \cdots \right) \end{aligned}$$

for all $z \in \mathbb{C}$.

Differentiate or integrate a known series (term by term) †

Differentiating ①, we have

$$\frac{1}{(1-z)^2} = 1 + 2z + 3z^2 + \cdots \qquad \text{for } |z| < 1$$

while integrating, we have

$$-\log(1-z) = z + \frac{1}{2} z^2 + \frac{1}{3} z^3 + \cdots \qquad \text{for } |z| < 1\,.$$

D. Zeros and Isolated Singularities

Let f be a complex function and a be a point in $\mathbb{C}$.

If f is defined and analytic at a, then we call a an analytic point of f. If in addition $f(a) = 0$, we call it a zero of f. If f is *not* defined at a but is defined and analytic *near* a, then we call a an isolated singularity of f.

† Can integrate term by term: if $\sum a_n (z-a)^n$ converges absolutely, then so does $\sum \frac{a_n}{n+1} (z-a)^{n+1}$ since

$$\left| \frac{a_n}{n+1} (z-a)^{n+1} \right| = |a_n (z-a)^n| \cdot \frac{|z-a|}{n+1}$$

and $|z-a|/(n+1) \longrightarrow 0$.

In this section we will assume that f is analytic in a region A except at *finitely many* isolated singularities $a_1, \ldots, a_s$ in A.† For *any* point $a \in A$, consider the Laurent expansion

$$f(z) \;=\; \sum_{n=-\infty}^{\infty} a_n(z-a)^n \quad \text{where} \quad a_n \;=\; \frac{1}{2\pi i}\int_{\partial D} \frac{f(\zeta)}{(\zeta-a)^{n+1}}\,d\zeta$$

for a suitably small disk D surrounding a, and set

$$o(f,a) \;=\; \min\{n \in \mathbb{Z} : a_n \neq 0\}$$

where by convention $\min(\varnothing) = -\infty$. Note that $\boxed{o(f,a) \geqslant 0 \iff \lim_{z\to a}(z-a)f(z) = 0}$.

Define the <u>principal</u> <u>part</u> and <u>residue</u> of f at a to be

$$\mathrm{PP}(f,a) \;=\; \sum_{n<0} a_n(z-a)^n \qquad \text{and} \qquad \mathrm{Res}(f,a) \;=\; a_{-1} \;=\; \frac{1}{2\pi i}\int_{\partial D} f(z)\,dz\,.$$

Note that when $o(f,a) \geqslant -1$ (which means f is either analytic at a or, in the terminology introduced below, f has a removable singularity or a simple pole at a) we can compute:

$$\boxed{\mathrm{Res}(f,a) \;=\; \lim_{z\to a}(z-a)f(z)}\,.$$

The importance of the residues at the isolated singularities $a_1, \ldots, a_s$ arises from the following observation: If γ is a <u>simple</u> closed contour in A that is null-homotopic in A and does not pass through any of the a_k's, then by the argument on page 24 we can compute

$$\int_\gamma f(z)\,dz \;=\; 2\pi i \sum_{k=1}^{s} \mathrm{Res}(f,a_k)\mathrm{I}(\gamma,a_k)\,.$$

In words, "the integral $\int_\gamma f$ is $2\pi i$ times the sum of the residues of f inside γ." This is the classical Residue Theorem, which has many useful applications; see the next chapter. The modern version of the theorem just allows for non-simple contours.

<u>Zeros</u> If f is analytic at $a \in A$, then $o(f,a) \geqslant 0$. The Laurent expansion is then just the Taylor expansion of f about a, and so of course $\mathrm{Res}(f,a) = 0$. If in addition a is a zero of f, and so $o(f,a) > 0$, then $o(f,a)$ (which is easily computed using derivatives; it is the smallest positive integer m such that $f^{(m)}(a) \neq 0$) is called the <u>order</u> or <u>multiplicity</u> of a. A zero of order 1 is also called a <u>simple</u> <u>zero</u>.

For example $\sin z$ has simple zeros at the integer multiples of π, whereas $\sin^2 z$ has zeros of order 2 at these points (exercise).

Note that near a zero a of order m, we can write

$$f(z) \;=\; (z-a)^m\varphi(z)$$

where $\varphi(z) = \sum_{n\geqslant 0} a_{n+k}(z-a)^n$ is analytic at a with $\varphi(a) \neq 0$. Conversely, if $f(z)$ can be so written, then by taking derivatives we see that f has a zero of order m at a.

<u>Isolated singularities</u> An isolated singularity a of f is classified as a

<u>removable</u> <u>singularity</u> , <u>pole</u> , or <u>essential</u> <u>singularity</u>

according to whether $o(f,a) \geqslant 0$, < 0 but finite, or $-\infty$. If a is a pole, then the positive integer $|o(f,a)|$ is called the <u>order</u> of a. A pole of order 1 is also called a <u>simple</u> <u>pole</u>. Sometimes a removable singularity is called a pole of order 0.

† This is restrictive; for example $\csc(1/z)$ on $\mathbb{C} - 0$ has isolated singularities at $1/n\pi$ for all $n \in \mathbb{Z}$.

For example $\sin z/z = 1 - z^2/3! + - \cdots$ has a removable singularity at 0, whereas $\cos z/z = 1/z - z/2! + - \cdots$ and $e^{1/z} = \cdots + 1/2!z^2 + 1/z + 1$ have, resp., a simple pole and an essential singularity at 0. We can then read off $\operatorname{Res}(\cos z/z, 0) = 1 = \operatorname{Res}(e^{1/z}, 0)$.

3.11 <u>Theorem</u> *If f has a removable singularity or a pole at a, then $o(f,a) = k$ if and only if $\exists\, \varphi$, analytic at a, with $\varphi(a) \neq 0$ and $f(z) = (z-a)^k \varphi(z)$ for all $z \neq a$ near a.*

<u>Proof</u> ($\Longrightarrow$) Referring to the Laurent series above, we take $\varphi(z) = \sum_{n \geqslant 0} a_{n+k}(z-a)^n$.

($\Longleftarrow$) Since $\varphi(z)$ is analytic at a, $f(z)/(z-a)^k$ is certainly bounded near a. Therefore, for sufficiently large M and sufficiently small r

$$\begin{aligned} |a_n| &= \left| \frac{1}{2\pi i} \int_{\partial D_r(a)} \frac{f(\zeta)}{(\zeta - a)^{n+1}}\, d\zeta \right| = \left| \frac{1}{2\pi i} \int_{\partial D_r(a)} \frac{f(\zeta)}{(\zeta-a)^k (\zeta - a)^{n+1-k}}\, d\zeta \right| \\ &< \frac{1}{2\pi}\, 2\pi r\, \frac{|M|}{r^{n+1-k}} = |M|\, r^{k-n} \end{aligned}$$

which goes to 0 as $r \to 0$, so $a_n = 0$ for $n < k$. Therefore $\varphi(z) = f(z)/(z-a)^k = a_k + a_{k+1}(z-a) + \cdots$. Letting $z \to a$ gives $a_k \neq 0$, and so $o(f,a) = k$. □

3.12 <u>Corollary</u> *If f and g are analytic in a region A except at a finite number of (possibly different) isolated singularities, then the same is true of fg and f/g, and*

$$o(fg, a) = o(f,a) + o(g,a) \qquad \textit{and} \qquad o(f/g, a) = o(f,a) - o(g,a)$$

for any $a \in A$ that is not an essential singularity of f or of g.

<u>Proof</u> Setting $n = o(f,a)$ and $d = o(g,a)$, we have by Theorem 3.11

$$f(z) = (z-a)^n \varphi(z) \qquad \text{and} \qquad g(z) = (z-a)^d \psi(z)$$

for z near a, with φ and ψ analytic and nonzero at a. Therefore

$$f(z)g(z) = (z-a)^{n+d} \varphi(z)\psi(z) \qquad \text{and} \qquad f(z)/g(z) = (z-a)^{n-d} \varphi(z)/\psi(z)$$

for $z \neq a$ near a, and the result follows since $\varphi\psi$ and φ/ψ are nonzero at a. □

3.13 <u>Casorati-Weierstrass Theorem</u> *If f has an essential singularity at a, then for any $b \in \mathbb{C}$, there exists a sequence $a_n \to a$ such that $f(a_n) \to b$.*[†]

<u>Proof</u> If not, then $g(z) = 1/(f(z) - b)$ is analytic and bounded on some punctured neighborhood of a. But then $(z-a)g(z) \to 0$ as $z \to a$, which implies that g has a removable singularity at a, and so $f(z) = b - 1/g(z)$ is either analytic or has a pole at a, by 3.12, a contradiction. □

This has a remarkable generalization, whose proof* we do not give here:

3.14 <u>Picard's Theorem</u> *If f has an essential singularity at a, then for any $b \in \mathbb{C}$, with one possible exception, there exists a sequence $a_n \to a$ such that $f(a_n) = b$ for all n.*

[†] In topological terms, this says that the image under f of any neighborhood of a is dense in $\mathbb{C}$.

4. Calculus of Residues

A. The Residue Theorem

On page 34 we stated the classical version of the theorem. Here is the modern version:

4.1 Residue Theorem *Let $f : A \to \mathbb{C}$ be analytic on $A - \{a_1, \ldots, a_s\}$ and γ be a null homotopic closed contour in A not passing through any of the a_k's. Then*

$$\int_\gamma f(z)\, dz \;=\; 2\pi i \sum_{k=1}^{s} \operatorname{Res}(f, a_k) \mathrm{I}(\gamma, a_k)\,.$$

Note that we do not assume that γ is simple. If it is, and is oriented counterclockwise, then this says that the integral of f around γ is equal to $2\pi i$ times the sum of the residues at the singularities of f inside γ, as previously noted (and proved).

Proof For each $k = 1, \ldots, s$, let p_k denote the principal part of f at a_k. Then p_k converges on $\mathbb{C} - a_k$, and uniformly on γ, by Theorem 3.8′, so

$$\int_\gamma p_k \;=\; 2\pi i \operatorname{Res}(f, a_k) \mathrm{I}(f, a_k).$$

Now the function $g = f - \sum p_k$ has a removable singularity at each a_j since it has a finite limit as $z \to a_j$ (namely $\ell_j = (f - p_j)(a_j) - \sum_{k \neq j} p_k(a_j)$) and so becomes analytic on A by setting $g(a_j)$ equal to that limit (by Corollary 2.13). Thus by Cauchy's Theorem

$$0 \;=\; \int g \;=\; \int f - \int \sum p_k \;=\; \int f - \sum \int p_k \;=\; \int f - 2\pi i \sum \operatorname{Res}(f, a_k) \mathrm{I}(f, a_k)$$

where $\sum = \sum_{k=1}^{s}$ and all the integrals are along γ. This completes the proof. □

B. Computing Residues

Recall that the residue of f at an isolated singularity a is the coefficient c_{-1} of $(z-a)^{-1}$ in the Laurent expansion $f(z) = \sum_{n=-\infty}^{\infty} c_n (z-a)^n$, and is given by integral formula

$$\operatorname{Res}(f, a) \;=\; \frac{1}{2\pi i} \int_\gamma f(\zeta)\, d\zeta$$

for any suitably small, positively oriented circle γ centered at a. Unfortunately this integral is generally hard to compute. Easier approaches:

① Check whether a is a removable singularity; if it is, then $\operatorname{Res}(f, a) = 0$. As noted above, a is removable if and only if $\lim_{z \to a} (z - a) f(z) = 0$, and this limit is sometimes easy to compute. For example $f(z) = (e^z - 1)/\sin z$ has a removable singularity (and thus zero residue) at 0 since $\lim_{z \to 0} z(e^z - 1)/\sin z = 0$.

② If $f = g/h$ with g and h analytic and $g(a) = 0$, then there is a formula for $\operatorname{Res}(f, a)$ in terms of the derivatives of g and h at a, obtained as follows. Suppose that

$$o(g, a) \;=\; r \quad \text{and} \quad o(h, a) \;=\; s\;, \quad \text{and set} \quad p \;=\; s - r$$

which equals $-o(f, a)$ by Corollary 3.12. (Recall that r and s are just the orders of the first nonvanishing derivatives of g and h at a.) Then a is a removable singularity if $p = 0$, in which case of course $\operatorname{Res}(f, a) = 0$.

If $p > 0$, then a is a pole of order p, and then the Taylor expansions

$$f(z) \;=\; \sum_{n\geqslant -p} f_n(z-a)^n \;,\quad \sum g(z) \;=\; \sum_{n\geqslant r} g_n(z-a)^n \quad\text{and}\quad h(z) \;=\; \sum_{n\geqslant s} h_n(z-a)^n,$$

where $g_n = g^{(n)}(a)/n!$ and $h_n = h^{(n)}(a)/n!$, yield

$$\left(f_{-p}(z-a)^{-p} + \cdots\right)\left(h_s(z-a)^s + \cdots\right) \;=\; g_r(z-a)^r + \cdots .$$

since $fh = g$. Expanding out, one can then solve for f_{-1} in terms of the g_n's and h_n's.

<u>Simple poles</u> Assume a is a simple pole (that is $p = 1$, and so $s = r + 1$). Then

$$\left(f_{-1}(z-a)^{-1} + \cdots\right)\left(h_s(z-a)^s + \cdots\right) \;=\; g_r(z-a)^r + \cdots .$$

Therefore $f_{-1}h_s = g_r$, and so

$$\operatorname{Res}(f,a) \;=\; f_{-1} \;=\; \frac{g_r}{h_s} \;=\; s\,\frac{g^{(r)}(a)}{h^{(s)}(a)},$$

or simply $g(a)/h'(a)$ when $r = 0$ and $s = 1$, e.g. for $1/h(z)$ when h has a simple zero at a.

<u>Exercises</u>: Show ⓐ $\operatorname{Res}(e^z/\sin z, 0) = 1$ ⓑ $\operatorname{Res}(z/(\cos z - 1), 0) = -2$.

<u>Higher order poles</u> To state the residue formula in general, we use matrices: We must solve the equation $\vec{f}H = \vec{g}$, where $\vec{f} = (f_{-p}\cdots f_{-1})$, $\vec{g} = (g_r \cdots g_{s-1})$ and H is the upper triangular $p\times p$ matrix with h_s's on the diagonal, h_{s+1}'s on the first superdiagonal, etc.:

$$\begin{pmatrix} f_{-p} & \cdots & f_{-1}\end{pmatrix}\begin{pmatrix} h_s & h_{s+1} & \cdots & h_{s+p-1} \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & h_{s+1} \\ 0 & \cdots & 0 & h_s \end{pmatrix} \;=\; \begin{pmatrix} g_r & \cdots & g_{s-1}\end{pmatrix}$$

By Cramer's rule $\operatorname{Res}(f,a) = f_{-1} = \det H_g \,/\, \det H = \det H_g \,/\, h_s^p$ where H_g is the matrix obtained from H by replacing the last row with $\vec{g}$.

For example, if a is a <u>double</u> <u>pole</u> (that is $p = 2$, and so $s = r + 2$) then

$$\operatorname{Res}(f,a) \;=\; f_{-1} \;=\; \frac{\det\begin{pmatrix} h_s & h_{s+1} \\ g_r & g_{r+1}\end{pmatrix}}{h_s^2} \;=\; \frac{g_{r+1}h_s - g_r h_{s+1}}{h_s^2}$$

<u>Exercises</u>: Show ⓐ $\operatorname{Res}(e^z/(z-1)^2, 1) = e$ ⓑ $\operatorname{Res}((e^z-1)/\sin^3 z, 0) = 1/2$.

In the next two sections, we discuss some applications of the residue theorem.

<u>C. The Argument Principle and Rouché's Theorem</u>

Let $f : A \to \mathbb{C}$ be analytic except at finitely many poles. Then f is said to be <u>meromorphic</u> in A. (Note that we do not allow essential singularities in A.) For simplicity we also assume that f has only finitely many zeros in A.

Now let γ is a null-homotopic closed curve in A that does not pass through any of the zeros or poles of f. Consider the integral

$$\frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}\,dz.$$

If γ is parametrized by $\gamma(t)$ for $t \in [a, b]$, then this integral can be written as

$$\frac{1}{2\pi i}\int_a^b \frac{f'(\gamma(t))\gamma'(t)}{f(\gamma(t))}\,dt \;=\; \frac{1}{2\pi i}\int_a^b \frac{(f\circ\gamma)'(t)}{(f\circ\gamma)(t)}\,dt = \frac{1}{2\pi i}\int_{f\circ\gamma}\frac{1}{z}\,dz \;=\; \mathrm{I}(f\circ\gamma, 0).$$

Thus it can be interpreted as the winding number of $f \circ \gamma$ about 0, or equivalently, the total change in the argument of f as γ is traversed.

4.2 <u>Argument Principle</u> *If f is meromorphic in A with zeros a_j and poles b_k (each repeated as many times as its order indicates) then*

$$I(f\circ\gamma, 0) \;=\; \textstyle\sum_j \mathrm{I}(\gamma, a_j) - \sum_k \mathrm{I}(\gamma, b_k)$$

for every null-homotopic closed curve that does not pass through any of the zeros or poles.

<u>Proof</u> If a is a zero of order m, then as noted on page 34, $f(z) = (z-a)^m\varphi(z)$, where φ is analytic and nonzero at a, and so $f'(a) = m(z-a)^{m-1}\varphi(z) + (z-a)^m\varphi'(z)$. Consequently $f'(z)/f(z) = m/(z-a) + \varphi'(z)/\varphi(z)$, and so f'/f has a simple pole at a with residue m. The same calculation for a pole b of order p shows that f'/f has a simple pole at b with residue $-p$. The result is now immediate from the residue theorem. □

Here is a useful application of this principle (which we state in a slightly unusual way):

4.3 <u>Rouché's Theorem</u> *Let f be analytic in a region A, and γ be a null-homotopic simple closed curve in A. If f can be written as the sum of two analytic functions g and h with $|g| > |h|$ on γ, then f and g have the same number of zeros (counting multiplicities) enclosed in γ.*

<u>Proof</u> By the hypothesis, f and g are zero-free on γ (since $|f| = |g+h| \geqslant |g| - |h| > 0$ and $|g| > |h| \geqslant 0$ on γ). Set $q = f/g$. Then since $|f - g| = |h| < |g|$ on γ, we see (dividing by $|g|$) that $|q-1| < 1$ on γ, and so the curve $q\circ\gamma$ lies in the open disc of radius 1 centered at 1. Thus $I(q\circ\gamma, 0) = 0$, and so by the argument principle $q = f/g$ has an equal number of zeros and poles inside γ (counting multiplicities, where we define the multiplicity of a pole to be its order). But the zeros of q are just the zeros of f, while the poles of q are the zeros of g, and the result follows. □

<u>Example</u> Let $f(z) = z^5 + 3z^2 + 7z - 2$.

1 How many roots does f have inside the unit circle C? Go for the largest single term: Since $|7z| = 7$ on C, while $|z^5 + 3z^2 - 2| \leqslant |z^5| + |3z^2| + 2 = 6$ on C, we see that f has the same number of roots inside C as does $7z$, namely one.

2 How many roots of f have modulus between 1 and 2? Noting that $|z^5| = 32$ on $2C$, while $|3z^2 + 7z - 2| \leqslant |3z^2| + |7z| + 2 = 28$ on $2C$, we see that f has the same number of roots inside $2C$ as z^5, namely 5. Therefore, using 1, we see that f has $4 = 5 - 1$ roots of modulus between 1 and 2.

D. Evaluation of Definite Integrals

The residue theorem provides an efficient tool for computing real definite integrals of many different types. We illustrate this technique, following Ahlfors, first for certain trigonometric integrals, and then for two types of improper integrals.

<u>Trigonometric Integrals</u> Consider any integral

$$\int_0^{2\pi} R(\cos\theta, \sin\theta)\, d\theta$$

where $R(x, y)$ is a rational function of two variables. The substitution $z = e^{i\theta}$, $dz = ie^{i\theta}d\theta$ (and so $d\theta = dz/iz$) transforms it into a contour integral around the unit circle C:

$$\oint f(z)\, dz \quad \text{where} \quad f(z) = R\left(\frac{z + z^{-1}}{2}, \frac{z - z^{-1}}{2i}\right)\frac{1}{iz}$$

Therefore by the residue theorem, the value of the original integral is just $2\pi i$ times the sum of the residues of $f(z)$ inside C.

<u>Example</u> Evaluate $I = \displaystyle\int_0^{\pi/2} \frac{1}{\sin^2\theta + 2}\, d\theta \;=\; \int_0^{2\pi} \frac{1/4}{\sin^2\theta + 2}\, d\theta$, for which

$$f(z) \;=\; \frac{1/4}{\left(\left(\dfrac{z - z^{-1}}{2i}\right)^2 + 2\right) iz} \;=\; \frac{iz}{z^4 - 10z^2 + 1} \;=:\; \frac{g(z)}{h(z)}.$$

Now $h(z)$ factors as $(z^2 - r)(z^2 - s)$ where $r = 5 - \sqrt{24}$ and of $s = 5 + \sqrt{24}$. Thus f has 4 simple poles at $\pm\sqrt{r}$ and $\pm\sqrt{s}$, and only the first two lie inside C. We compute

$$\operatorname{Res}(f, \pm\sqrt{r}) \;=\; \frac{g(\pm\sqrt{r})}{h'(\pm\sqrt{r})} \;=\; \frac{\pm i\sqrt{r}}{\pm 2\sqrt{r}(r - s)} \;=\; \frac{i}{2(r - s)} \;=\; \frac{-i}{4\sqrt{24}}.$$

Therefore $I = 2\pi i(-i/2\sqrt{24}) = \pi/\sqrt{24}$.

For homework you are asked to evaluate $I = \displaystyle\int_0^{\pi} \frac{1}{\cos\theta + 2}\, d\theta$, which is easier.

<u>Improper Integrals</u> Let $R(x) = P(x)/Q(x)$, where P and Q are polynomials of degree p and q, and Q is nonzero on $\mathbb{R}$ (so necessarily of even degree). We consider two types of improper integrals:

$$I_1 \;=\; \int_{-\infty}^{\infty} R(x)\, dx \qquad \text{where } q \geqslant p + 2$$

$$I_2 \;=\; \int_{-\infty}^{\infty} R(x)e^{i\omega x}\, dx \qquad \text{where } q \geqslant p + 1^{\dagger} \text{ and } \omega \text{ is a positive real number}$$

By definition, these are the limits of the corresponding finite integrals from $-r$ to s, as r and s tend *independently* to ∞. In fact I_1 can be equivalently be defined as the limit of the integral from $-r$ to r as $r \to \infty$ (since the half improper integrals from 0 to $\pm\infty$ both converge). This is not the case for I_2.

① For I_1 the procedure is to integrate the analogous complex function $R(z)$ over a contour C_r consisting of the directed line segment $[-r, r]$ along the real axis, followed by

† The integral I_2 is the value at ω of the <u>Fourier</u> <u>transform</u> of R, usually denoted

$$\widehat{R}(\omega) \;=\; \int_{-\infty}^{\infty} R(x)e^{i\omega x}\, dx \;=\; \int_{-\infty}^{\infty} R(x)\cos(\omega x)\, dx + i\int_{-\infty}^{\infty} R(x)\sin(\omega x)\, dx\,.$$

This transform is of great importance in PDE's (solutions to the heat equation), theoretical physics, quantum mechanics, etc.

the semicircular arc A_r from r to $-r$ in the upper half plane. For r large enough, C_r will contain all the poles of R (which are among the zeros of Q) and so

$$\int_{C_r} R(z)\,dz \;=\; 2\pi i\,\mathrm{Res}_+(R)$$

where $\mathrm{Res}_+(R)$ is the sum of the residues of R at all its poles in the upper half plane. Also if r is large enough, then for some constant c, $|R(z)| < c/r^2$ for all z on A_r (since $q \geqslant p+2$) and so we can estimate

$$\left|\int_{A_r} R(z)\,dz\right| \;\leqslant\; \frac{c\pi r}{r^2} \;=\; \frac{c\pi}{r}.$$

Since this goes to 0 as $r \to \infty$, it follows that

$$I_1 \;=\; \lim_{r\to\infty}\int_{C_r} R(z)\,dz \;=\; 2\pi i\,\mathrm{Res}_+(R).$$

Example Evaluate $\displaystyle\int_{-\infty}^{\infty}\frac{P(x)}{Q(x)}\,dx$ where $P(x) = x^2 - x + 2$ and $Q(x) = x^4 + 10x^2 + 9$. Note that $Q(x) = (x^2+1)(x^2+9)$, and so it has roots $\pm i$ and $\pm 3i$. By the discussion above, the answer is $2\pi i$ times the sum of the residues of R at i and $3i$, which are

$$\mathrm{Res}(R,i) \;=\; \frac{P(i)}{Q'(i)} \;=\; \frac{1-i}{16i} \qquad\text{and}\qquad \mathrm{Res}(R,3i) \;=\; \frac{P(3i)}{Q'(3i)} \;=\; \frac{-7-3i}{-48i}$$

and so the integral equals $2\pi i\,((1-i)/16i + (7+3i)/48i) = 5\pi/12$.

For homework you are asked to evaluate the analogous integral when $P(x) = x^2 + x + 1$ and $Q(x) = x^4 + 5x + 4$.

2 For I_2, note that $|e^{i\omega z}| = e^{-\omega y}$ (for $z = x + iy$) is bounded in the upper half plane, so the same estimates as above show that

$$I_2 \;=\; \int_{-\infty}^{\infty} R(x)e^{i\omega x}\,dx \;=\; 2\pi i\;\mathrm{Res}_+(R(z)e^{i\omega z})$$

provided $q \geqslant p+2$. In fact the same result holds when $q = p+1$, but in this case, as noted above, we must compute

$$\lim_{r,s\to\infty}\int_{-r}^{s} R(x)e^{ix}\,dx\,.$$

For this computation, it is more convenient to use the rectangular contours

$$\gamma_{r,s,y} \;=\; [-r,s] + [s,s+iy] + [s+iy,-r+iy] + [-r+iy,-r] \quad (\text{where } r,s,y>0).$$

Since $q = p+1$, there is a constant C such that $|R(z)| < C/|z|$ for $|z|$ sufficiently large. Hence the integral along the right vertical side $[s, s+iy]$ can be estimated

$$\left|\int_{[s,s+iy]} R(z)e^{i\omega z}\,dz\right| \;=\; \left|\int_0^y R(s+it)e^{i\omega(s+it)}i\,dt\right| \;\leqslant\; \frac{C}{s}\int_0^y e^{-\omega t}\,dt \;\leqslant\; \frac{C}{\omega s}(1-e^{-\omega y})$$

which is less than $C/\omega s$ since $y > 0$. Similarly the integral along the left vertical side is bounded in absolute value by $C/\omega r$. The integral along the top is easily seen to be bounded in absolute value by $Ce^{-\omega y}(r+s)/y$, which for fixed r and s tends to 0 as $y \to \infty$. Therefore

$$|I_2 - 2\pi i\,\mathrm{Res}_+(R(z)e^{i\omega z})| \;<\; C(1/\omega r + 1/\omega s).$$

for sufficiently large r and s. Letting r and s go to ∞, we see that $I_2 = 2\pi i\,\mathrm{Res}_+(R(z)e^{iz})$.

5. Conformal Maps

Let $A \subset \mathbb{C}$ be open and connected and $f : A \to \mathbb{C}$ be continuously differentiable (when viewed as a real function) with df_a nonsingular at some $a \in A$. For our present purposes, we define the notion of "conformality" slightly differently (a priori) than we did in §1.D.

Definition The function f is conformal at a if it is analytic at a with $f'(a) \neq 0$.

We now introduce two notions of a more geometric nature that are closely related to this analytic definition. Consider any parametrized contour γ passing through a, say $\gamma(0) = a$, with $\gamma'(0) \neq 0$. Thus $\gamma'(0)$ can be viewed as the velocity vector of a particle moving through a at time zero.

We will use the classical expressions for functions in terms of variables, using subscripts to denote derivatives. Thus if $z = x + iy = \gamma(t)$ and $w = u + iv = f(z)$, then we will write z_t for $\gamma'(t)$, w_z for $f'(z)$ (if it exists), w_t for $(f \circ \gamma)'(t)$, w_x for $\partial f/\partial x$, etc. The hypotheses $\gamma'(0) \neq 0$ and df_a nonsingular show that $z_t \neq 0$ at 0 and $w_z \neq 0$ at a, so by the chain rule, $w_t \neq 0$ at 0 as well.

In this notation, the two Cauchy-Riemann equations $u_x = v_y$, $u_y = -v_x$ can be written in complex form as the single equation $w_y = iw_x$.

Definition Given $w = f(z)$, $z = \gamma(t)$ with $\gamma(0) = a$ as above, we say the function f

a rotates uniformly at a if $\arg(w_t/z_t)$ is independent of γ at $t = 0$.

b dilates uniformly at a if $|w_t/z_t|$ is independent of γ at $t = 0$.

It is understood that both z_t and w_t are nonzero at $t = 0$.

5.1 Conformal Criterion *If f is conformal at a, then it rotates and dilates uniformly at a. Conversely,* (a) *if f rotates uniformly at a, then it is conformal at a, and* (b) *if f dilates uniformly at a, then either f or $\bar{f}$ is conformal at a.*

Proof The first statement is just the "Conformal Mapping Theorem" (Corollary 1.7) proved using the chain rule on page 12.

For (a), assume that f rotates uniformly at a. Then by the chain rule

$$w_t \;=\; w_x x_t + w_y y_t \;=\; \tfrac{1}{2}(w_x - iw_y)(x_t + iy_t) + \tfrac{1}{2}(w_x + iw_y)(x_t - iy_t).$$

Setting $c = \frac{1}{2}(w_x - iw_y)$ and $r = \frac{1}{2}(w_x + iw_y)$, we see that $w_t/z_t = c + r\, z_t/\bar{z}_t$ which describes a circle of radius r (centered at c) as z_t varies. This has constant argument if and only if $r = 0$, that is if and only if $w_y = iw_x$, which is just the Cauchy-Riemann equation. Therefore f is analytic at a, and the chain rule implies that $f'(a)$ is nonzero, since z_t and w_t are nonzero by hypothesis.

For (b), assume that f dilates uniformly at a. Then as above we see that either $r = 0$ or $c = 0$. In the former case we conclude that f is conformal at a, as above, while in the latter case we have $w_y = -iw_x$, and so $\bar{f}$ is conformal at a. $\square$

NOTES IN PROGRESS ...