# CALCULUS II (Math 102)

Text: *Essential Calculus 2/e* by James Stewart

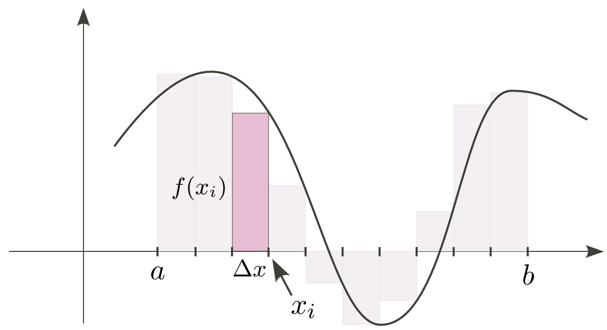
**Calculus**: The study of real functions $f$, and their derivatives $f'$ and integrals $\int f$.

Let $f : [a, b] \to \mathbb{R}$ be continuous.

**Definition** The definite integral of $f$ from $a$ to $b$ is

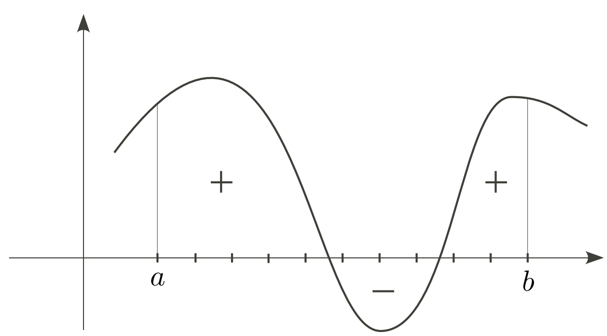
$$\int_a^b f \;=\; \int_a^b f(x)\,dx \;:=\; \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\,\Delta x$$

where $\Delta x = (b-a)/n$ and $x_i = a + i\,\Delta x$. Note: $x$ is a "dummy" variable, replaced in many physical applications by time $t$, and soi $\int_a^b f = \int_a^b f(t)\,dt$.




The base of the shaded rectangle is $\Delta x$, and $f(x_i)$ is its height.

**Physical Meaning** distance traveled (during time $a \le t \le b$, where $f(t) =$ velocity at time $t$)
**Geometric Meaning** signed area (between the graph of $f$ and $[a, b] \subset x$-axis)




**Fundamental Formula of Calculus** (FFC)

$$\int_a^b f(x)\,dx \;=\; F(b) - F(a) \qquad \Big(\text{also written } F(x)\Big|_a^b\Big)$$

for any antiderivative $F$ of $f$ on $[a, b]$, meaning $F' = f$ on $[a, b]$.† Since any two such antiderivatives differ by a constant (a consequence of the Mean Value Theorem (MVT)) it follows that any antiderivative can be used; the constants will cancel!

† Antiderivatives are also called primitives or indefinite integrals .

This wonderful formula, which reduces the problem of computing definite integrals to the problem of computing antiderivatives, follows from the following remarkable result:

**Fundamental Theorem of Calculus** (FTC) The function $F : [a, b] \to \mathbb{R}$ defined by

$$F(x) \;=\; \int_a^x f(x)\,dx \;=\; \begin{cases} \text{signed area between the graph of } f \\ \text{and the interval } [a, x] \text{ on the } x\text{-axis} \end{cases}$$

is an antiderivative of $f$. In other words, $\dfrac{d}{dx}\displaystyle\int_a^x f(x)\,dx = f(x)$.

This theorem can be understood geometrically using the area interpretation of the definite integral, and proved rigorously using – once again – the MVT. The FFC follows:

$$F(b) - F(a) \;=\; \int_a^b f(x)\,dx - \int_a^a f(x)\,dx \;=\; \int_a^b f(x)\,dx.$$

Because of this theorem, we often simply write $\int f(x)\,dx$ to denote the antiderivative(s) of $f$. Thus for example $\int x\,dx = \frac{1}{2}x^2 + C$ and $\int x^2\,dx = \frac{1}{3}x^3 + C$.

**Techniques of Integration**

- **Power Rule** $\displaystyle\int x^p\,dx \;=\; \frac{x^{p+1}}{p+1}$

In words: to integrate a power, raise the power by 1, and then divide by the new power. This follows from the power rule for derivatives (differentiate the RHS to verify it) so we also refer to it as the backward power rule.

- **Sum and Difference Rule** $\displaystyle\int (f(x) \pm g(x))\,dx \;=\; \int f(x)\,dx \pm \int g(x)\,dx$

- **Integration by Parts** (backward product rule)

$$\int f(x)g(x)\,dx = f(x)G(x) - \int f'(x)G(x)\,dx.$$

where $G$ is any antiderivative of $g$ (differentiate both sides to verify) More on this later.

- **Integration by Substitution** (backward chain rule)

If the substitution $u = u(x)$ and $du = u'(x)dx$ converts

$$\int f(x)\,dx \qquad \text{into} \qquad \int g(u)\,du$$

(that is, if $f(x) = g(u(x))\,u'(x)$) then the first integral can be computed by evaluating the second, and then plugging $u(x)$ back in for $u$. The key here is to find a $u$ for which the new integral is easier to evaluate than the old one. In particular, *if $u(x)$ and $u'(x)$ both appear in $f(x)$, then substituting $u = u(x)$ is potentially productive.*

For definite integrals, one can avoid plugging back in by substituting for the bounds:

$$\boxed{\int_a^b f(x)\,dx \;=\; \int_{u(a)}^{u(b)} g(u)\,du}$$

## 4.5 Integration by Substitution

**Examples** ① $\int x\sqrt{x^2+1}\,dx \;=\; \int x(x^2+1)^{1/2}\,dx.$

Substitute $u = x^2+1,\; du = 2x\,dx$ (so $\frac{1}{2}\,du = x\,dx$) to get

$$\frac{1}{2}\int u^{1/2}\,du \;=\; \frac{1}{2}\,\frac{2}{3}\,u^{3/2}+C \;=\; \frac{1}{3}\,u^{3/2}+C$$

and, plugging $x^2+1$ back in for $u$, $\int x(x^2+1)^{1/2}\,dx \;=\; \frac{1}{3}(x^2+1)^{3/2}+C\,.$

② $\int_0^1 x\sqrt{x^2+1}\,dx.$

With the same substitution, and noting that $u(0)=1$ and $u(1)=2$, we get

$$\frac{1}{2}\int_1^2 u^{1/2}\,du \;=\; \frac{1}{3}\,u^{3/2}\Big|_1^2 \;=\; \frac{1}{3}(\sqrt{8}-1)\,.$$

③ $\int x^3\sqrt{x^2+1}\,dx.$

With the same substitution, there will be an extra $x^2 = u-1$, so get

$$\frac{1}{2}\int (u-1)u^{1/2}\,du \;=\; \frac{1}{2}\int (u^{3/2}-u^{1/2})\,du \;=\; \frac{1}{2}(\frac{2}{5}\,u^{5/2}-\frac{2}{3}\,u^{3/2})+C$$

and, plugging $x^2+1$ back in for $u$,

$$\int x\sqrt{x^2+1}\,dx \;=\; \frac{1}{5}(x^2+1)^{5/2}-\frac{1}{3}(x^2+1)^{3/2}+C\,.$$

④ $\int x^2\sqrt{x^2+1}\,dx.$

With the same substitution, there will be now be an extra $x = (u-1)^{1/2}$, so get

$$\frac{1}{2}\int (u-1)^{1/2}u^{1/2}\,du \;=\; \frac{1}{2}\int (u^2-u)^{1/2}\,du$$

and its not clear that the substitution has helped. We're stuck, at least for now.

⑤ $\int \sec^3 x \tan x\;dx.$

Substitute $u = \sec x,\; du = \sec x \tan x$ to get

$$\int u^2\,du \;=\; \frac{1}{3}u^3+C \;=\; \frac{1}{3}\sec^3 x + C\,.$$

What happens if you substitute $u = \tan x$? End up with

$$\int u(u^2+1)^{1/2}\,du \;=\; \frac{1}{3}(u^2+1)^{3/2}+C$$

from example (1) above, which equals $\frac{1}{3}\sec^3 x + C$ since $\tan^2 x + 1 = \sec^2 x$.

⑥ (like HW 4.5 (50)) $\displaystyle\int_0^2 x^2\sqrt{64-x^6}\,dx$.

If you substitute $u = 64 - x^6$ this becomes the integral of $\sqrt{u/(64-u)}$, up to a constant. Ugh! But if you substitute $u = x^3$ it becomes

$$\frac{1}{3}\int_0^8 \sqrt{64-u^2}\,du\,.$$

Now observe that the graph of $v = \sqrt{64-u^2}$ in the $uv$-plane is the upper half of the circle $u^2 + v^2 = 8^2$. Thus the last integral can be interpreted as the area $16\pi$ of a quarter of that circle, and so the original integral has value $16\pi/3$.

## 7.1 Areas Between Curves

Let $R$ be a bounded region in the plane. Choose an axis, parametrized say by $t$. For each value of $t$, let $L(t)$ denote the corresponding cross-sectional length of $R$, and suppose that $L(t)$ is positive for all $t$ in $(a, b)$, and zero outside of $[a, b]$. Then the area of $R$ is given by

$$A \;=\; \int_a^b L(t)\,dt$$

Special case : $R$ is bounded by curves

I) the $x$-axis, the graph of $f(x) > 0$ and the vertical lines $x = a$ and $x = b$ : $A = \displaystyle\int_a^b f(x)\,dx$

II) the graph of $y = u(x)$ and of $y = \ell(x) \le u(x)$ : $A = \displaystyle\int_a^b (u(x) - \ell(x))\,dx$

- Find the limits $a, b$ of integration by solving $f(x) = g(x)$.
- If you don't know which of $u$ or $\ell$ is larger, take the absolute value at the end.

**Examples** Find the areas of the regions bounded by

① $y = x$ and $y = x^2$ : The curves intersect where $x = x^2$, that is, where $x = 0$ and $x = 1$. Clearly $x$ is larger than $x^2$ along $[0, 1]$, and so

$$A \;=\; \int_0^1 (x - x^2)\,dx \;=\; (\frac{1}{2}x^2 - \frac{1}{3}x^3)\Big|_0^1 \;=\; \frac{1}{6}$$

② $y = x^2 - 3$, $y = 5 - x^2$ : The curves intersect where $x^2 - 3 = 5 - x^2$, that is, where $2x^2 = 8$, so $x = -2$ and $x = 2$. The region is clearly symmetric about the $y$-axis, so we can (to simplify the math) compute twice the area along the interval $[0, 2]$. But suppose we don't realize that the second curve is on top. Then

$$A \;=\; \left|2\int_0^2 ((x^2-3) - (5-x^2))\,dx\right| \;=\; \left|\int_0^2 (2x^2 - 8)\,dx\right| \;=\; \left|(\frac{2}{3}x^3 - 8x)\Big|_0^2\right| \;=\; \frac{32}{3}.$$

③ $x + 1 = 2(y-2)^2$ and $x + 6y = 7$ (integrate with respect to $y$)

## 5.1 Inverse Functions

In one-variable calculus we study functions $f : X \to \mathbb{R}$ whose domain $X$ is a subset of the real numbers $\mathbb{R}$. The function assigns a real number $f(x)$ to each number $x$ in $X$.

The domain is an indispensable part of the function. If not specified, it is understood to be the set of all numbers $x$ for which $f(x)$ makes sense. For example the domain of the function $f(x) = 1/(x-1)$ is the set of all real numbers not equal to 1.

The range of $f : X \to \mathbb{R}$ is the set of all the numbers $f(x)$ that arise as $x$ ranges over $X$. For example the range of $f(x) = 1/(x-1)$ is the set of nonzero real numbers.

**Examples** Find the domain and range of

① the sine function : The domain is $\mathbb{R}$, while the range is the closed interval $[-1, 1]$.

② the function $f(x) = \dfrac{x^2+1}{x^2-1}$ : The domain is the set of all real numbers $\neq \pm 1$.

The range is the set of all real numbers $\leq -1$ or $> 1$. This takes a little work to verify.

**Definition** A function $f : X \to \mathbb{R}$ is one-to-one (also written as 1-1) provided it *never* takes on the same value twice, that is $f(x_1) \neq f(x_2)$ whenever $x_1 \neq x_2$.

Equivalently, this says that every horizontal line intersects the graph of $f$ in at most one point. This is the horizontal line test for checking if $f$ is 1-1.†

Here is a useful sufficient condition for a differentiable function defined on an interval to be 1-1; the interval can be closed, half open, or open (which includes $\mathbb{R} = (-\infty, \infty)$):

**Theorem** (Derivative Test) *If $X$ is an interval, and $f : X \to \mathbb{R}$ is differentiable with $f'(x)$ always of the same sign (either always positive or always negative), then $f$ is* 1-1.

This follows from the Mean Value Theorem, which implies that $f$ is monotonic (i.e. either increasing or decreasing) on $X$, and so certainly 1-1.

**Examples** Which of the following functions are 1-1?

① the sine function : this is not 1-1, for example $\sin(0) = \sin(\pi)$

② $f(x) = x^3 + 2x + 1$ : the derivative is $3x^2 + 2$, which is positive for all $x$, so $f$ is 1-1

③ $T(t)$ = temperature in this room as a function of time $t$ : this is surely not 1-1

**Definition** If $f : X \to \mathbb{R}$ is a 1-1 function, then it has an inverse function

$$f^{-1} : Y \longrightarrow \mathbb{R} \qquad \text{(where } Y \text{ is the range of } f\text{)}$$

sending each $y$ in $Y$ to the unique $x$ in $X$ for which $f(x) = y$. Thus the domain of $f^{-1}$ is the range of $f$, and the range of $f^{-1}$ is the domain of $f$. We often simply write

$$X \underset{f^{-1}}{\overset{f}{\rightleftarrows}} Y$$

† Compare this with the vertical line test for checking if a curve in the plane is the graph of a function.

Clearly $f$ does not have an inverse function if it is not 1-1. Thus

$$f \text{ is 1-1} \iff f^{-1} \text{ exists} \iff f \text{ satisfies the horizontal line test}$$

How to try to find a formula for $f^{-1}$

- Set $y = f(x)$
- Try to solve for $x$ in terms of $y \rightsquigarrow x = f^{-1}(y)$
- Swap $x$ and $y$ to give the conventional form $y = f^{-1}(x)$.

**Examples** Find a formula for the inverse functions of the following 1-1 functions:

1. $f(x) = x^3 : y = x^3 \Longrightarrow x = \sqrt[3]{y}$. Therefore $f^{-1}(x) = \sqrt[3]{x}$
2. $f(x) = x^3 + 2x + 1$ : stuck
3. $f(x) = \dfrac{2x+1}{3x-5} : y = \dfrac{2x+1}{3x-5} \Longrightarrow y(3x-5) = 2x+1 \Longrightarrow (3y-2)x = 1+5y$
   $\Longrightarrow x = \dfrac{1+5y}{3y-2}$. Therefore $f^{-1}(x) = \dfrac{1+5x}{3x-2}$

**Properties** The inverse function $f^{-1} : Y \to \mathbb{R}$ of a 1-1 function $f : X \to \mathbb{R}$ satisfies:

- $y = f(x)$ if and only if $f^{-1}(y) = x$
- $f^{-1}(f(x))$ and $f(f^{-1}(y)) = y$, for all $x \in X$ and $y \in Y$
- the graph of $f^{-1}$ is the reflection of the graph of $f$ through the line $y = x$

The last property shows geometrically (though it's hard to prove analytically) that:

- If $f$ is continuous, then so is $f^{-1}$
- If $f$ is differentiable with nonzero derivative everywhere, then so is $f^{-1}$

This final fact is the first half of the very useful:

**Inverse Function Theorem** (IFT) *Let $f : X \to \mathbb{R}$ be a* 1-1 *differentiable function with $f'(x) \neq 0$ for all $x$ in $X$. Then the inverse function $f^{-1} : Y \to \mathbb{R}$ is differentiable, with derivative $(f^{-1})'(f(x)) = 1/f'(x)$.*

Proof Differentiate $f^{-1}(f(x)) = x$ (using the chain rule) to get $(f^{-1})'(f(x))\, f'(x) = 1$. Now divide by $f'(x)$ and substitute $y$ for $f(x)$. $\square$

**Remark** Practically speaking, the IFT shows

(a) If you want to compute $(f^{-1})'(b)$ for some particular number $b$, try to find $a$ for which $f(a) = b$. Then $(f^{-1})'(b) = 1/f'(a)$.

(b) If you want a general formula for $(f^{-1})'(x)$, write $y = f^{-1}(x)$ (reversing the roles of $x$ and $y$). Then $x = f(y)$. Suppose that we know how to write $f'(y) = dx/dy$ in terms of $x$ (e.g. replacing $y$ by $f^{-1}(x)$). Then we get an explicit formula for

$$\frac{dy}{dx} = \frac{1}{dx/dy}.$$

We will give many examples of (b) when we discuss inverse trig functions, and later the exponential function, but first we illustrate (a):

**Examples** (1) Given $f(x) = x^3 + 2x + 1$, find $(f^{-1})'(4)$. Solution: noting that $f(1) = 4$, we have $(f^{-1})'(4) = 1/f'(1) = 1/(3x^2 + 2)|_{x=1} = 1/5$.

(2) (like HW 5.1 (44)) Let $g$ be the inverse function of a differentiable 1-1 function $f$, and let $h = g^2$. If $f(3) = 4$ and $f'(3) = 5$, find $h'(4)$. Solution: By the chain rule, $h'(4) = 2g(4)g'(4) = 2 \cdot 3/f'(3) = 6/5$.

## 5.6 Inverse Trigonometric Functions

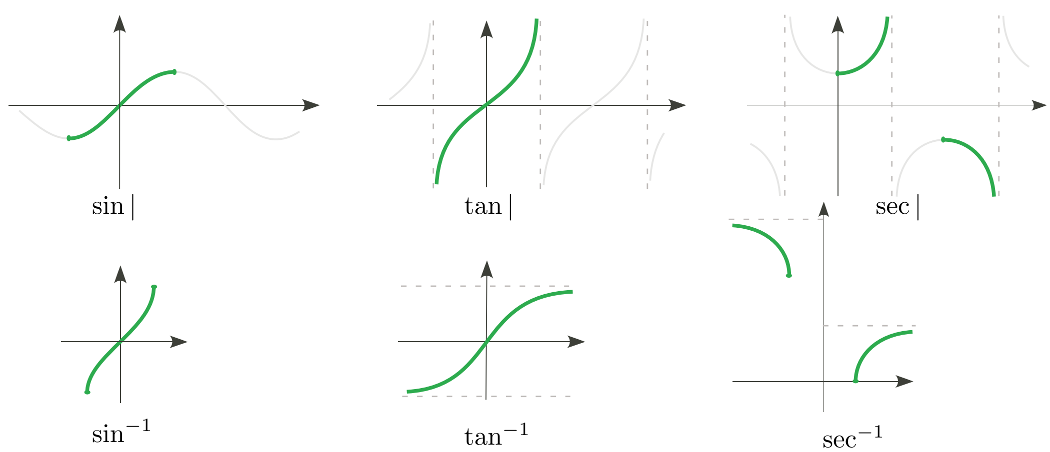
First observe that none of the trig functions sin, tan, sec, ... are 1-1 (look at their graphs) so we must restrict their domains in order to make them 1-1. These restrictions are denoted $\sin|$, $\tan|$, $\sec|$, ... (shown in green below) and their inverses are what we simply call $\sin^{-1}$, $\tan^{-1}$, $\sec^{-1}$, ...:

**Definition**

$$[-\tfrac{\pi}{2}, \tfrac{\pi}{2}] \xrightleftharpoons[\sin^{-1}]{\sin|} [-1, 1] \qquad\qquad (-\tfrac{\pi}{2}, \tfrac{\pi}{2}) \xrightleftharpoons[\tan^{-1}]{\tan|} (-\infty, \infty)$$

and

$$[0, \tfrac{\pi}{2}) \cup [\pi, \tfrac{3\pi}{2}) \xrightleftharpoons[\sec^{-1}]{\sec|} (-\infty, -1] \cup [1, \infty)$$




**Formulas**

$$\text{(1)}\quad \frac{d}{dx}\sin^{-1}x = \frac{1}{\sqrt{1-x^2}} \qquad \text{and} \qquad \sin^{-1}x = \int \frac{1}{\sqrt{1-x^2}}\,dx$$

$$\text{(2)}\quad \frac{d}{dx}\tan^{-1}x = \frac{1}{x^2+1} \qquad \text{and} \qquad \tan^{-1}x = \int \frac{1}{x^2+1}\,dx$$

$$\text{(3)}\quad \frac{d}{dx}\sec^{-1}x = \frac{1}{x\sqrt{x^2-1}} \qquad \text{and} \qquad \sec^{-1}x = \int \frac{1}{x\sqrt{x^2-1}}\,dx$$

**Proofs** (illustrating (b) above)

① $y = \sin^{-1} x$. Thus $x = \sin y$ and so

$$\frac{dy}{dx} = \frac{1}{dx/dy} = \frac{1}{\cos y} = \frac{1}{\sqrt{1-\sin^2 x}} = \frac{1}{\sqrt{1-x^2}}.$$

② $y = \tan^{-1} x$. Thus $x = \tan y$ and so

$$\frac{dy}{dx} = \frac{1}{dx/dy} = \frac{1}{\sec^2 y} = \frac{1}{\tan^2 x + 1} = \frac{1}{x^2+1}.$$

③ Homework 5.6 (#14)

**Examples** Using the formulas above (and the chain rule / substitution)

① If $c$ is constant, then $\displaystyle \frac{d}{dx}\tan^{-1}(c/x) = \frac{1}{(c/x)^2+1} \cdot \frac{-c}{x^2} = -\frac{c}{c^2+x^2}$

② $$\frac{d}{dx}\tan^{-1}(\sin^{-1}(x^3)) = \frac{1}{(\sin^{-1}(x^3))^2+1} \cdot \frac{3x^2}{\sqrt{1-x^6}}$$

③ $$\int \frac{1}{x\sqrt{4x^2-9}}\,dx = \int \frac{1}{x\sqrt{9(4x^2/9-1)}}\,dx = \int \frac{1}{3x\sqrt{(2x/3)^2-1}}\,dx.$$

Substituting $u = 2x/3$, $du = (2/3)dx$ gives

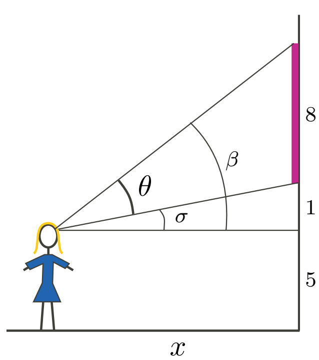
$$\int \frac{1}{3u\sqrt{u^2-1}}\,du = \frac{1}{3}\sec^{-1} u + C = \frac{1}{3}\sec^{-1}(\frac{2x}{3}) + C.$$

④ (like HW 5.6 (38)) You're 5 ft tall, and in a museum looking at a 8 ft tall painting hanging on the wall with its top at 14 ft above the floor. At what distance $x$ away from the wall should you stand to have the maximum viewing angle?

Solution: The viewing angle $\theta$ is given as a function of $x$ by

$$\theta = \beta - \sigma = \tan^{-1}(9/x) - \tan^{-1}(1/x)$$

as seen by drawing a picture.




We compute

$$\frac{d\theta}{dx} = -\frac{9}{81+x^2} + \frac{1}{1+x^2}$$

which is zero when the two fractions are equal, that is, when $9 + 9x^2 = 81 + x^2$, or $x = 3$. This is evidently a maximum, and so the optimal distance is 3 feet.

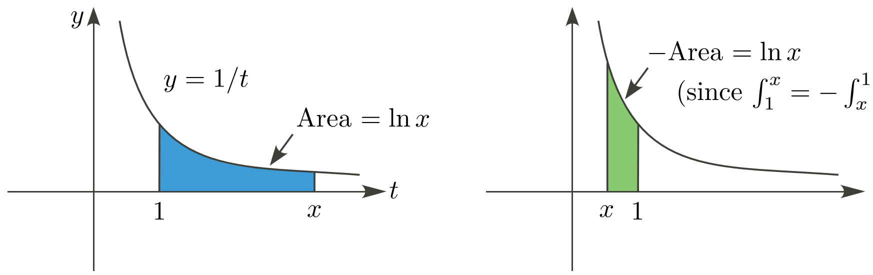
## 5.2 The Natural Logarithm

The natural logarithm function, denoted ln, is defined for all positive real numbers $x$ by

$$\ln x \;=\; \int_1^x \frac{1}{t}\,dt$$

(we use $t$ for the integration variable since $x$ is being used as the indep. variable of ln). Note: ln is not defined for $x \le 0$; its domain is the set $\mathbb{R}_+$ of all positive reals.

Geometrically, $\ln x$ represents a signed area under the graph of $y = 1/t$, as shown below:




Thus $\ln x$ is negative for $x < 1$ and positive for $x > 1$, and clearly zero when $x = 1$ (using either the integral definition or the area interpretation).

We will show below that the range of ln is all of $\mathbb{R}$. This is not obvious. For example, why should one be able to achieve an arbitrarily large blue area by choosing $x$ large enough? (Think about the analogous question for the function $f(x) = \int_1^x (1/t^2)\,dt$)

### The derivative of ln

By the Fundamental Theorem, we compute

$$\frac{d}{dx}\ln x \;=\; \frac{d}{dx}\int_1^x \frac{1}{t}\,dt \;=\; \frac{1}{x}.$$

which holds for $x > 0$, where $\ln x$ is defined. There is an analogous formula for $x < 0$, when using the chain rule we compute $\ln(-x)\,' = -1/(-x) = 1/x$. Thus, in fact

$$\frac{d}{dx}\ln|x| \;=\; \frac{1}{x}$$

for all nonzero $x$, positive or negative.

The corresponding integration formula is

$$\int \frac{1}{x}\,dx \;=\; \ln|x| + C.$$

Below, we will give explicit examples that use these derivative and integral formulas.

But first ...

### Algebraic laws of ln

For any $a, b \in \mathbb{R}_+$ and any rational number $r$

(a) $\ln(ab) \;=\; \ln a + \ln b$ (b) $\ln(a/b) = \ln a - \ln b$ (c) $\ln(b^r) \;=\; r\ln b$

Qualitatively, this says that the logarithm converts products into sums, quotients into differences, and powers into products, which explains its great historical significance.

Proofs

(a) Set $f(x) = \ln(xb)$. Then $f'(x) = b/xb = 1/x$, by the chain rule, so $f(x) = \ln x + C$ for some constant $C$. Substituting $x = 1$ shows that $C = \ln b$. Therefore $\ln(ab) = f(a) = \ln a + \ln b$ as claimed.

(b) From (a) we have $\ln a = \ln((a/b)b) = \ln(a/b) + \ln b$, and the result follows.

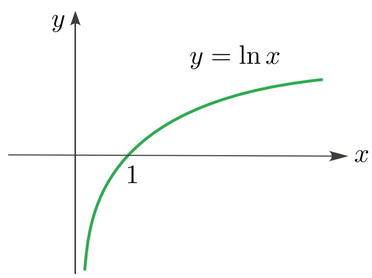
(c) Set $g(x) = \ln(x^r)$. Then $g'(x) = rs^{r-1}/x^r = r/x$, so $g(x) = \ln x + C$. Substituting $x = 1$ shows that $C = 0$. Therefore $\ln(b^r) = f(b) = r \ln b$ as claimed.

## The graph of ln

Monotonicity  $\ln' x = 1/x > 0$ for all $x > 0$, so $\ln x$ is an increasing function.†

Concavity  $\ln'' x = (1/x)' = -1/x^2 < 0$ for all $x$, so the graph of ln is concave down.

Thus, recalling that $\ln 1 = 0$, we can give a rough sketch of the graph:




Asymptotic behavior

The logarithm function increases without bound as $x \to \infty$, and decreases without bound as $x \to 0$, meaning

$$\lim_{x\to\infty} \ln x = +\infty \qquad \text{and} \qquad \lim_{x\to 0} \ln x = -\infty.$$

Indeed $\ln 2^{\pm n} = \pm n \ln 2$ can be made as large in absolute value as we want by choosing $n$ sufficiently large. Since ln is continuous – indeed differentiable – it follows from the Intermediate Value Theorem that the range of ln is all of $\mathbb{R}$, as claimed above.

Later we will show – using L'Hôpital's Rule for evaluating limits of quotients – that $\ln x$ "grows slower" than any power $x^n$ (for $n = 1, 2, \ldots$), meaning

$$\lim_{x\to\infty} \ln x / x^n = 0 = \lim_{x\to 0} x^n \ln x.$$

## Differentiation of logarithmic functions

Examples

(1) Find an equation of the tangent line to $y = \sin(2 \ln x)$ at $(1, 0)$: The slope is $y'(1) = \cos(2 \ln x)(2/x)|_{x=1} = 2 \cos 0 = 2$, so the equation of the line is $y = 2x - 2$.

(2) Implicit Differentiation  Find $y'$ if $y$ is defined implicitly by $y = \ln(x + y^2)$: Differentiating both sides w.r.t. $x$ gives $y' = (1 + 2yy')/(x + y^2)$. To solve for $y'$, first cross multiply, giving $(x + y^2)y' = 1 + 2yy'$, and then collect the $y'$ terms, giving $(x + y^2 - 2y)y' = 1$. Therefore $y' = 1/(x + y^2 - 2y)$.

† and therefore 1-1; we will discuss its inverse function "exp" in the next section

③ Logarithmic Differentiation A special technique used to compute the derivative of complicated functions based on the philosophy that the logarithm simplifies the algebra. If $y = f(x)$, then by the chain rule, $(\ln y)' = y'/y$, and so $y' = y(\ln y)'$. Thus

$$f'(x) \;=\; f(x) \cdot (\ln f(x))'.$$

for any function $f(x)$.

Examples Find $y'$ for the given $y$.

ⓐ $y = \sqrt{\dfrac{(x^2+1)^3 \tan x}{\sqrt{x}}} \implies (\ln y)' = y \cdot \frac{1}{2}(3\ln(x^2+1) + \ln\tan x - \frac{1}{2}\ln x)'$, so

$$y' \;=\; y(\ln y)' \;=\; \sqrt{\frac{(x^2+1)^3 \tan x}{\sqrt{x}}}\left(\frac{3x}{x^2+1} + \frac{\sec^2 x}{2\tan x} - \frac{3}{4x}\right).$$

ⓑ $y = x^x \implies y' = y(\ln y)' = x^x(x\ln x)' = x^x(1\ln x + x(1/x)) = x^x(\ln x + 1)$.†

ⓒ $y = f(x)^{g(x)}$ (a good exercise to test your understanding) †

## Integration with logarithms

Recall that $\int (1/x)\,dx = \ln|x|$ (suppressing the "$+C$" for convenience). Therefore

$$\int \frac{f'(x)}{f(x)}\,dx \;=\; \ln|f(x)|$$

for any function $f(x)$, as seen by substituting $u = f(x)$. For example

$$\int \cot x\,dx \;=\; \int \frac{\cos x}{\sin x}\,dx \;=\; \ln|\sin x|$$

$$\int \tan x\,dx \;=\; \int \frac{\sin x}{\cos x}\,dx \;=\; -\ln|\cos x| \;=\; \ln|\sec x|.$$

More Examples

① $\displaystyle\int \frac{1}{x\ln x}\,dx = \int \frac{1/x}{\ln x}\,dx = \ln(\ln x)$.

② $\displaystyle\int \sec x\,dx$. Trick: multiply the integrand by $1 = (\sec x + \tan x)/(\sec x + \tan x)$,

$$\int \sec x\,dx \;=\; \int \frac{\sec^2 x + \sec x\tan x}{\sec x + \tan x}\,dx \;=\; \ln|\sec x + \tan x|\,.$$

③ $\displaystyle\int \frac{3(\ln x)^2}{x}\,dx$. Substituting $u = \ln x$, we get $\displaystyle\int 3u^2\,du = u^3 + C = \ln^3 x$.

④ $\displaystyle\int_1^2 \frac{3(\ln x)^2}{x}\,dx$. The substitution in ③ leads to $\displaystyle\int_0^{\ln 2} u^2\,du \;=\; u^3\Big|_0^{\ln 2} \;=\; (\ln 2)^3$.

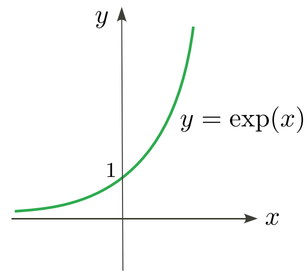
† In ⓑ and ⓒ we allow arbitrary real exponents, so that $x^x$ and more generally $f(x)^{g(x)}$ should make sense when $x$ or $g(x)$ are real. This will be justified when we discuss exponential functions below.

## 5.3 The Exponential Function

The natural exponential function, denoted exp, is the inverse function of the natural logarithm function:

$$\mathbb{R}_+ \underset{\exp}{\overset{\ln}{\rightleftarrows}} \mathbb{R}$$

so exp has domain $\mathbb{R}$ and range $\mathbb{R}_+$. Note that $\exp(0) = 1$ since $\ln 1 = 0$. We sketch the graph of $y = \exp(x)$ by reflecting the graph of $y = \ln x$ about the line $y = x$:



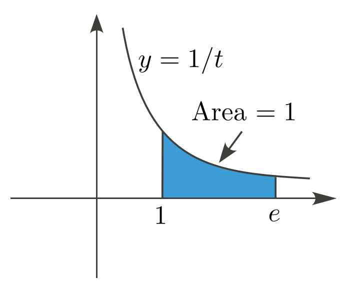

The asymptotic behavior of ln translates into the corresponding behavior of exp:

$$\lim_{x\to\infty} \exp(x) \;=\; \infty \qquad \text{and} \qquad \lim_{x\to-\infty} \exp(x) \;=\; 0\,.$$

and as for logarithms, L'Hôpital's rule will show that exponential functions of $x$ "grow faster" than any power $x^n$ $(n = 1, 2, \dots)$, i.e. $\lim_{x\to\infty} x^n / \exp(x) = 0 = \lim_{x\to-\infty} x^n \exp(x)$.

### The number e

Define $e := \exp(1)$. Thus $e$ is defined by the equation $\ln e = 1$, and so is the number for which $\int_1^e (1/t)\, dt = 1$.




From the definition of the integral as a limit of Riemann sums, one can (with some work) approximate $e$ to any desired accuracy:

$$e \;\approx\; 2.71828182845904523536028747135266249775724709369995957496697$$

In fact $e$ is an irrational number. This can be shown using calculus, but we won't take the time to do so here.

Recall that if $r$ is rational, then $\ln(e^r) = r \ln e = r$, and so $e^r = \exp(r)$. We now define

$$e^x \;:=\; \exp(x) \qquad \text{or more generally} \qquad b^x \;:=\; \exp(x \ln b) \;=\; e^{x \ln b}$$

for any real number $x$ and any $b > 0$. (You should check that this agrees with the usual definition when $x$ is rational.) Taking the logarithm of both sides shows that the formula

$$\ln(b^x) \;=\; x \ln b$$

now holds for all $x$.

**A limit formula for e**

It is a remarkably useful fact (used in financial applications) that

$$e \;=\; \lim_{n\to\infty}(1+1/n)^n\,.$$

To see why this is true, we compute the derivative of $\ln x$ at $x=1$ from the definition, which we also know from the formula $\ln' x = 1/x$ must equal 1:

$$1 \;=\; \ln'(1) \;=\; \lim_{h\to 0}\frac{\ln(1+h)-\ln 1}{h} \;=\; \lim_{h\to 0}\ln\left((1+h)^{1/h}\right) \;=\; \ln\left(\lim_{h\to 0}(1+h)^{1/h}\right)$$

since ln is a continuous function. Taking exp of both sides yields the desired formula:

$$e \;=\; \lim_{h\to 0}(1+h)^{1/h} \;=\; \lim_{n\to\infty}(1+1/n)^n\,.^{\dagger}$$

**The derivative of exp**

Set $y=\exp(x)=e^x$. Then $\ln y = x$, and so

$$\frac{d}{dx}e^x \;=\; \frac{dy}{dx} \;=\; \frac{1}{dx/dy} \;=\; \frac{1}{1/y} \;=\; y \;=\; e^x\,.$$

Thus $e^x$ is its own derivative! And so also its own integral: $\int e^x\,dx = e^x$.

**Algebraic laws of exp**

The laws of logarithms: $\ln(ab)=\ln a+\ln b$, $\ln(a/b)=\ln a-\ln b$ and $\ln(b^x) \;=\; x\ln b$, translate into the familiar laws of exponents:

$$\text{(a)}\;\; e^{p+q} \;=\; e^pe^q \qquad \text{(b)}\;\; e^{p-q} \;=\; e^p/e^q \qquad \text{(c)}\;\; (e^q)^x \;=\; e^{qx}$$

for any real numbers $p$, $q$ and $x$. Indeed the latter follow from the former by setting $p=\ln a$ and $q=\ln b$, and then applying exp. And a quick check shows that these laws hold with $e$ replaced by any positive real number $b$:

$$\text{(a)}\;\; b^{p+q} \;=\; b^pb^q \qquad \text{(b)}\;\; b^{p-q} \;=\; b^p/b^q \qquad \text{(c)}\;\; (b^q)^x \;=\; b^{qx}$$

as well as the familiar law (d) $(bc)^x = b^xc^x$ for any positive $b$ and $c$.

**Differentiation and integration with exponential functions**

Examples

(1) Find an equation of the tangent line to $y = e^{x^2+1}\sin x$ at $(0,0)$: The slope is $y'(0) = (2xe^{x^2+1}\sin x + e^{x^2+1}\cos x)|_{x=0} = e$, so the equation of the line is $y = ex$.

(2) Find the absolute maximum value of the function $f(x)=2x-e^x$. We compute $f'(x)=2-e^x$. This is 0 when $e^x=2$, i.e. when $x=\ln 2$, $>0$ to the left of $\ln 2$ and $<0$ to the right. Therefore $f$ has an absolute maximum at $x=\ln 2\approx .7$, with value $f(\ln 2)=2\ln 2-e^{\ln 2}=2\ln 2-2\approx -.6$.

(3) Evaluate $\displaystyle\int_0^1 e^x\sqrt{1+e^x}\,dx$. Substituting $u=1+e^x$ we obtain

$$\int_0^1 e^x\sqrt{1+e^x}\,dx \;=\; \int_2^{1+e}\sqrt{u}\,du \;=\; \tfrac{2}{3}u^{3/2}\Big|_2^{1+e} \;=\; \tfrac{2}{3}((1+e)^{3/2}-\sqrt{8}) \;\approx\; 2.89.$$

† This last limit converges very slowly – for example $(1+1/400)^{400} = 2.715$ to 3 decimal places, while the correct value of $e$ to 3 decimal places is 2.718 – so it cannot be used as a practical way to compute $e$.

## 5.4 General Logarithmic and Exponential Functions

For any fixed $b > 0$ define the new functions

$$\log_b x \;=\; \ln x / \ln b \qquad \text{and} \qquad \exp_b(x) \;=\; b^x \;=\; e^{x \ln b}.$$

Thus the $\log_b x$ is obtained from $\ln x$ by dividing by the constant $\ln b$, and $\exp_b(x)$ is obtained by multiplying $x$ by $\ln b$ and then applying exp. These are inverse functions

$$\log_b(b^x) \;=\; x \qquad \text{and} \qquad b^{\log_b x} = x$$

as is readily verified.†

### Algebraic laws

As noted above, $b^x$ satisfies the familiar algebraic laws, and a similar check shows that $\log_b$ does as well:

$$\text{(a) } \log_b xy \;=\; \log_b x + \log_b y \qquad \text{(b) } \log_b x/y = \log_b x - \log_b y \qquad \text{(c) } \log_b x^p \;=\; p \log_b x$$

### Derivatives and Integrals

The derivatives of $\log_b$ and $\exp_b$ are easily computed:

$$\frac{d}{dx} \log_b x \;=\; \frac{1}{x \ln b} \qquad \text{and} \qquad \frac{d}{dx} b^x \;=\; b^x \cdot \ln b.$$

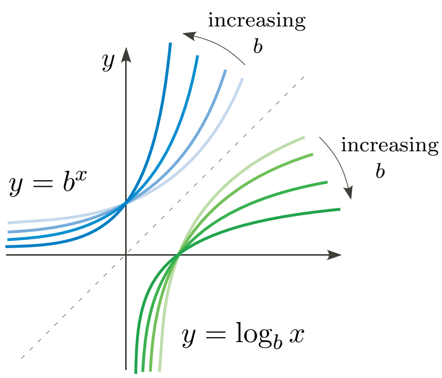
For example, the second formula follows from the chain rule: $(b^x)' = (e^{x \ln b})' = e^{x \ln b}(x \ln b)'$ $= b^x \cdot \ln b$, and immediately yields the integration formula:

$$\int b^x \, dx \;=\; b^x / \ln b + C.$$

Example $\displaystyle\int 4x \cdot 5^{x^2} \, dx \;=\; \int 2 \cdot 5^u \, du \quad (\text{where } u = x^2) \;=\; 2 \cdot 5^u / \ln 5 \;=\; 2 \cdot 5^{x^2} / \ln 5.$

### Graphs

These are discussed in some detail in the text. The basic principal is that the larger the base $b$, the slower $\log_b$ grows, and the faster $b^x$ grows.




† $\log_b(b^x) = \ln(b^x)/\ln b = x \ln b / \ln b = x$ and $b^{\log_b x} = e^{(\ln x/\ln b)\ln b} = e^{\ln x} = x$.

## 5.5 Applications: Exponential Growth and Decay

### Population Growth

Let $P(t)$ be the population at time $t$ (of some city/animal tribe/bacteria culture, etc.), and set $P_0 = P(0)$. With no environmental restrictions, we can reasonably assume the growth rate $P'(t)$ is proportional to $P(t)$, i.e. $P(t)$ satisfies the IVP ("initial value problem")

$$\boxed{P'(t) \;=\; k\,P(t)\ ,\ P(0) = P_0}$$

for some $k > 0$ (generally not given; it depends on the nature of the population). One obvious solution to these equations is

$$\boxed{P(t) \;=\; P_0\,e^{kt}} \qquad \text{(check this by differentiating).}$$

In fact any solution must be of this form.†

Examples 1 A population was 1 million last year and 3 million this year. What do you expect it to be next year?

Solution In the notation above $P_0 = P(0) = 3$, so $P(t) = 3e^{kt}$ for some $k$. But we also know $P(-1) = 1$, which means $3e^{-k} = 1$, or $k = \ln 3$. Thus $P(t) = 3e^{t\ln 3} = 3^{t+1}$, and so the population next year will be $P(1) = 3^{1+1} = 9$ million.

2 If a population doubles in 10 years, how long will it take to triple?

Solution We are given that $P(10) = 2P_0 = P_0e^{10k}$, and so $e^{10k} = 2$. Thus $k = \ln 2/10$, so $P(t) = P_0e^{t\ln 2/10} = P_0 2^{t/10}$. We want to find $t$ for which $P(t) = 3P_0$, that is, $2^{t/10} = 3$. Solving we find $t = 10\log_2 3 \approx 15.8$ years.

### Radioactive Decay

The mass $m(t)$ of a physical substance at time $t$, with initial mass $m_0 = m(0)$, typically decays at a rate $m'(t)$ proportional to $m(t)$, with some proportionality constant $k > 0$ depending on the substance. So it satisfies the IVP

$$\boxed{m'(t) \;=\; -k\,m(t)\ ,\ m(0) = m_0} \quad \text{with solution} \quad \boxed{m(t) \;=\; m_0\,e^{-kt}}.$$

Often one knows the half life $h$ of the substance, defined by $m(h) = m_0/2 = m_0e^{-kh}$, and so $k$ and $h$ are related by the equation $kh = \ln 2$. It follows that $k = \ln 2^{-1/h}$, and so

$$\boxed{m(t) \;=\; m_0\,2^{-t/h}}.$$

Example (Carbon dating of fossils – Willard Libby 1949) Living tissue has 2 isotopes of carbon: C12 (stable) and C14 (radioactive, with a half life $h = 5500$ years). When it dies, the C12 remains and C14 decays.

For example if the C14 has decayed to 20% of the original, then the age $t$ of the fossil satisfies $m(t) = m_0 2^{-t/h} = m_0/5$, so $-t/h = \log_2 1/5 \implies t = 5500\log_2 5 \approx 12,270$ years. In general, if mass $m$ remains, then the fossil is $t = h\log_2(m_0/m)$ years old.

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† If $Q(t)$ is another solution, then $(P/Q)' = (P'Q - PQ')/Q^2 = ((kP)Q - P(kQ))/Q^2 = 0$, so $P/Q$ is some constant $C$. But $P(0)/Q(0) = P_0/P_0 = 1$, so $C = 1$, whence $Q = P$.

**Cooling**

Newton's Law of Cooling says that the temperature $T(t)$ at time $t$ of a hot liquid sitting in a room cools according to the IVP

$$\boxed{T'(t) = -k(T(t) - T_r)\ ,\ T(0) = T_0}$$

for some $k > 0$ (depending on the liquid), where $T_r$ is the room temperature. The solution

$$\boxed{T(t) \ = \ T_r + (T_0 - T_r)e^{-kt}}$$

is obtained as follows: Normalize the scale so the room temperature is zero, i.e. set $\widetilde{T}(t) = T(t) - T_r$. The equations become $\widetilde{T}'(t) = -k\widetilde{T}(t)\,,\ \widetilde{T}(0) = T_0 - T_r$. Solve this as above: $\widetilde{T}(t) = (T_0 - T_r)e^{-kt}$, and then add back in $T_r$.

Example (Coffee problem) If you pour your cup now, but plan to drink it later, should you add milk now or later for a hotter cup?

Solution Let $C$ and $M$ be the temperature now of the coffee and the milk (in the frig), and $p$ be the portion the total liquid that is milk. For simplicity suppose $T_r = 0$, so $C > 0$ and $M < 0$. The temperatures in the two scenarios are

- (add now) $T^n \ = \ T_0e^{-kt} \ = \ (pM + (1-p)C)e^{-kt}$
- (add later) $T^\ell \ = \ pM + (1-p)Ce^{-kt}$.

Subtracting we see that $T^n - T^\ell = pM\,(e^{-kt} - 1) > 0$, since both $pM$ and $e^{-kt} - 1$ are negative. So add the milk now!

**Compound Interest**

An investment of $P_0$ dollars at interest rate $r$ (i.e. $100r$ percent per year) compounded continuously obeys the IVP

$$\boxed{P'(t) \ = \ r\,P(t)\ ,\ P(0) \ = \ P_0} \quad \text{with solution} \quad \boxed{P(t) \ = \ P_0\,e^{rt}}.$$

Alternate derivation: With yearly compounding, $P_0$ will grow to $P_0(1+r)^t$ in $t$ years. If compounded $n$ times/yr, it'll grow to $P_0(1 + r/n)^{nt}$, and if compounded continuously, to

$$P_0 \lim_{n\to\infty}(1 + r/n)^{nt} \ = \ P_0 \lim_{n\to\infty}(1 + 1/(n/r))^{(n/r)rt} \ = \ P_0e^{rt}.$$

Example How long will it take for an investment of \$1000 invested at an interest rate of .05 (i.e. $r = .05$, compounded continuously) to triple? What is the equivalent annual interest rate during that time, i.e. at what rate $s$, compounded annually, would the investment have tripled in that same amount of time?

Solution For the investment to triple, we must find $t$ for which $1000e^{.05t} = 3000$, so $t = 20\ln 3 \approx 22$ years. The equivalent annual rate $s$ satisfies $1000(1+s)^t = 3000$, so $(1+s)^t = e^{.05t}$, or $s = e^{.05} - 1 \approx 5.13\%$.

**A slightly harder problem** (food for thought)

The snow plow problem: It started snowing one morning, and continued to snow steadily all day. A snowplow, working steadily (i.e. removing a fixed volume of snow each hour) plowed 2 miles of the road between noon and 1 PM, and 1 more mile by 2 PM. At what time did it start snowing?

## 6.1 Integration by Parts

Recall the product rule $(FG)' = F'G + FG'$. Integrating (i.e. antidifferentiating) both sides yields $FG = \int F'G + \int FG$, or rearranging terms:

$$\int FG' = FG - \int F'G.$$

This is integration by parts, or "the backwards product rule". In applying this method, one must decide how to split the integrand as a product $FG'$. The goal is to choose $F$ and $G'$ so that $F'G$ is simpler to integrate than $FG'$.

**Remark** Here are two other ways to view this technique:

- Write the integrand as a product $Fg$ of two functions $F$ and $g$ (the "parts") where $F$ is easy to differentiate and $g$ is easy to integrate. Then setting $f = F'$ and $G = \int g$,

$$\int Fg = FG - \int fG.$$

In words, the integral of $F$ times $g$ is ($F$ times the integral of $g$) minus (the integral of (the derivative of $F$ times the integral of $g$)).

- Using the standard notation for integrals using the differential $dx$, write the integral as $\int udv$ for some $u = u(x)$, $v = v(x)$, where $dv = v'(x)dx$ as usual. Then

$$\int u\,dv = uv - \int v\,du.$$

**Examples** Compute $\int h(x)\,dx$ where $h(x) =$

1. $xe^x$ $(F = x,\ G' = e^x)$
2. $\ln x$ $(F = \ln x,\ G' = 1)$
3. $x^2 \sin x$ $(F = x^2,\ G' = \sin x)^\dagger$
4. $\sin^2 x$ $(F = \sin x,\ G' = \sin x)$
5. $\sec^3 x$ $(F = \sec x,\ G' = \sec^2 x)$
6. $e^x \sin 2x$ $(F = \sin 2x,\ G' = e^x)$

1 becomes $xe^x - \int e^x\,dx = e^x(x-1)$ , 2 becomes $x\ln x - \int 1\,dx = x(\ln x - 1)$

3 becomes $-x^2\cos x + 2\int x\cos x\,dx$. Integrating by parts again with $F = x$, $G' = \cos x$ leads to $-x^2\cos x + 2x\sin x - 2\int \sin x\,dx = (2 - x^2)\cos x + 2x\sin x$.

4 becomes $-\sin x\cos x + \int \cos^2 x\,dx = -\sin x\cos x + \int (1 - \sin^2 x)\,dx$ and so rearranging terms we see that $2\int \sin^2 x\,dx = x - \sin x\cos x$, so $\int \sin^2 x\,dx = \frac{1}{2}(x - \sin x\cos x)$.

5 becomes $\sec x\tan x - \int \sec x\tan^2 x\,dx = \sec x\tan x - \int(\sec^3 x - \sec x)\,dx$. Rearranging terms gives $2\int \sec^3 x\,dx = \sec x\tan x + \int \sec x\,dx = \sec x\tan x + \ln|\sec x + \tan x|$, so $\int \sec^3 x\,dx = \frac{1}{2}(\sec x\tan x + \ln|\sec x + \tan x|)$.

6 becomes $e^x\sin 2x - 2\int e^x\cos 2x\,dx = e^x\sin 2x - 2e^x\cos 2xe^x - 4\int e^x\sin 2x\,dx$. Now, rearranging terms gives $\int 5e^x\sin 2x\,dx = e^x(\sin 2x - 2\cos 2x)$, so $\int e^x\sin 2x\,dx = \frac{1}{5}e^x(\sin 2x - 2\cos 2x)$.

† Apply the method twice

**Reduction Formulas** Examples:

① $I_n := \int x^n e^x\,dx \;=\; x^n e^x - n\int x^{n-1}e^x\,dx$ (using $F = x^n$, $G' = e^x$). Thus

$$\boxed{I_n \;=\; x^n e^x - nI_{n-1}}.$$

Apply this repeatedly, noting that $I_0 = e^x$, to compute $I_n$. For example

$$\begin{aligned} I_3 \;&=\; \int x^3 e^x\,dx \;=\; x^3e^x - 3I_2 \;=\; x^3e^x - 3(x^2e^x - 2I_1) \\ &=\; x^3e^x - 3(x^2e^x - 2(xe^x - I_0)) \;=\; e^x(x^3 - 3x^2 + 6x - 6). \end{aligned}$$

② Here is another way to think about this iterative process: For any function $F$ and any integer $n \geq 0$, let $F^{(n)}$ denote the $n$th derivative of $F$, and $F^{(-n)}$ denote the $n$th integral of $F$. In particular $F^{(0)} = F$. Now given two functions $F$ and $G$, arrange their derivatives and integrals in two adjacent rows, as shown:

$$\begin{array}{ccccccc} \cdots & F^{(-2)} & F^{(-1)} & F^{(0)} & F^{(1)} & F^{(2)} & \cdots \\ \cdots & G^{(2)} & G^{(1)} & G^{(0)} & G^{(-1)} & G^{(-2)} & \cdots \end{array}$$

Set $I_n = \int F^{(n)}G^{(1-n)}$, so in particular $I_0 = \int FG'$ and $I_1 = \int F'G$. The integration by parts formula implies that $I_n = F^{(n)}G^{(-n)} - I_{n+1}$ for every $n$. Starting with $I_0$, and repeating, then yields the formula $I_0 = F^{(0)}G^{(0)} - F^{(1)}G^{(-1)} + F^{(2)}G^{(-2)} - + \cdots$, that is:

$$\int FG' \;=\; FG - F'(\textstyle\int G) + F''(\iint G) - F'''(\iiint G) + \cdots$$

Note that the series ends if $F$ is a polynomial (so some derivative of $F$ is zero). This gives a quick alternative derivation of the integral $\int x^3e^x\,dx$ in ①.

③ $I_n := \int \sin^n = -\sin^{n-1}\cos + (n-1)\int \sin^{n-2}\cos^2$ (using $F = \sin^{n-1}$, $G' = \sin$). Substituting $1 - \sin^2$ for $\cos^2$ and rearranging gives $nI_n = -\sin^{n-1}\cos + (n-1)I_{n-2}$, so

$$\boxed{I_n \;=\; \tfrac{1}{n}\left((n-1)I_{n-2} - \sin^{n-1}x\,\cos x\right)}.$$

Since $I_0 = x$ and $I_1 = -\cos x$, this allows one to compute $I_n$ for all $n$. For example $I_2 = \int \sin^2 x\,dx = \frac{1}{2}(x - \sin x\cos x)$, and $I_3 = \int \sin^3 x\,dx = \frac{1}{3}(-2\cos x - \sin^2 x\cos x)$

## 6.2 Trigonometric Substitution

Integrals involving square roots of quadratic expressions in $x$ often succumb to a trig substitution, in which $x$ is taken to be a trig function of the new variable $u$ (so effectively we are substituting $u =$ an inverse trig function of $x$).

Here's the scheme:

| **Expression Before** | **Substitution** | **Expression After** |
|---|---|---|
| $a^2 - x^2$ | $x = a\sin u,\ dx = a\cos u\,du$ | $a^2\cos^2 u$ |
| $a^2 + x^2$ | $x = a\tan u,\ dx = a\sec^2 u\,du$ | $a^2\sec^2 u$ |
| $x^2 - a^2$ | $x = a\sec u,\ dx = a\sec u\tan u\,du$ | $a^2\tan^2 u$ |

**Examples** Compute $\int f(x)\,dx$ where $f(x) =$

① $1/\sqrt{4-x^2} \rightsquigarrow \int 1$

② $x/\sqrt{4-x^2} \rightsquigarrow \int \sin$

③ $x^2/\sqrt{4-x^2} \rightsquigarrow \int \sin^2$

④ $1/\sqrt{9x^2+4} \rightsquigarrow \int \sec$

⑤ $1/(9x^2+4)^{5/2} \rightsquigarrow \int \cos^3$

⑥ $\sqrt{1+x^2} \rightsquigarrow \int \sec^3$

⑦ $x^2\sqrt{25-x^2} \rightsquigarrow \int \sin^2\cos^2$

⑧ $x^3\sqrt{25-x^2} \rightsquigarrow \int \sin^3\cos^2$

**Integrating trig expressions**

- $\cos^p x$

  a) $p$ odd : substitute $u = \sin x$, $\cos^2 = 1-u^2 \rightsquigarrow$ polynomial in $u$

  b) $p$ even : apply trig identity $\cos^2 x = \frac{1}{2}(1+\cos 2x)$ repeatedly / or use parts

  Similarly for $\sin^q x$, $q$ odd ($u = \cos x$); $q$ even (use $\sin^2 x = \frac{1}{2}(1-\cos 2x)$ or parts)

- $\sec^p x$

  a) $p$ even : substitute $u = \tan x$, $\sec^2 = 1+u^2 \rightsquigarrow$ polynomial in $u$

  b) $p$ odd : parts

- $\cos^p x \sin^q x$

  a) $p$ odd : substitute $u = \sin x$ as above (similarly for $q$ odd, substitute $u = \cos x$)

  b) $p$ and $q$ even : apply trig identity $\cos^2 x = \frac{1}{2}(1+\cos 2x)$ repeatedly / or parts

- $\sec^p x \tan^q x$

  a) $p$ even : $u = \tan x$

  b) $q$ odd : $u = \sec x \tan x$

**Examples** ① $\int \sin^2 x\,dx$ ② $\int \sin^3 x\,dx$ ③ $\int \sin^2 x\cos^2 x\,dx$ ④ $\int \sin^3 x\cos^2 x\,dx$

**The last resort** (Weierstrass, mid 19th century)

To integrate any rational function of $\sin x$ and $\cos x$, substitute

$$u \;=\; \tan(x/2) \qquad \Longrightarrow \qquad x = 2\tan^{-1} u\,, \quad dx = \frac{2}{1+u^2}\,du.$$

Then

$$\sin x \;=\; 2\sin(x/2)\cos(x/2) \;=\; \frac{2\tan(x/2)}{\sec^2(x/2)} \;=\; \frac{2u}{1+u^2}$$

$$\cos x \;=\; \cos^2(x/2)-\sin^2(x/2) \;=\; \frac{1-\tan^2(x/2)}{\sec^2(x/2)} \;=\; \frac{1-u^2}{1+u^2}$$

which leads to a rational function of $u$ (discussed in the next section).

**Example** $\displaystyle\int \frac{1}{1+\cos x}\,dx \;=\; \int \frac{1}{1+\frac{1-u^2}{1+u^2}}\,\frac{2}{1+u^2}\,du \;=\; \int du \;=\; u \;=\; \tan(x/2).^{\dagger}$

$^{\dagger}$ Note that the derivative of $\tan(x/2) = \frac{1}{2}\sec^2(x/2)$. To see why this is equal to $1/(1+\cos x)$, use the half angle trig identity: $\cos x = 2\cos^2(x/2) - 1$.

## 6.3 Integration of Rational Functions

A rational function is a quotient of polynomials.† For example

$$f(x) \;=\; \frac{x+1}{x^3-2x^2+x} \qquad \text{and} \qquad g(x) \;=\; \frac{x^3+2x+1}{x^2+1}$$

are rational functions, while $h(x) = \sqrt{x^2+1}$ and $\sin(x^2-1)$ are not. A rational function is called proper if its numerator has a smaller degree (meaning its highest exponent) than its denominator. Thus $f$ is proper while $g$ is not.

**Goal**: Learn how to integrate all rational functions – assuming we can first carry out the appropriate algebra of factoring and dividing polynomials, and of decomposing rational functions into simpler ones (see below).

**Facts** (1) Every polynomial can be factored into quadratic and linear polynomials.

(2) Every *improper* rational function $f(x) = n(x)/d(x)$ is a sum of a polynomial and a *proper* rational function: Divide $n(x)$ by $d(x)$ to get a quotient $q(x)$, with remainder $r(x)$ satisfying $\deg r < \deg d$. Then $f = q + r/d$, and $r/d$ is proper.

(3) partial fractions Every *proper* rational function $r(x)/d(x)$ is a sum of rational functions of the form $p(x)/q(x)^k$ with $\deg p < \deg q \leq 2$. Here the $q(x)$'s are the factors of the denominator $d(x)$, and $k$ can take on any value $\leq$ the multiplicity of $q(x)$ in $d(x)$.

We assume you know how to do (1) (which is hard in general) and (2) (which is easy), and we'll learn through examples how to do (3) (after factoring both numerator and denominator using (1)). Since we know how to integrate polynomials, this reduces the problem of integrating any rational function to the following special cases:

$$\text{(a)}\ \frac{1}{(x-a)^k} \qquad\qquad \text{(b)}\ \frac{1}{(x^2+bx+c)^k} \qquad\qquad \text{(b)}\ \frac{2x}{(x^2+bx+c)^k}$$

But (a) is easy (use the power rule, except when $k=1$ where the integral is $\ln|x-a|$) and (c) reduces to (b) by the following trick:

$$\frac{2x}{\Box^k} \;=\; \frac{2x+b-b}{\Box^k} \;=\; \frac{2x+b}{\Box^k} - b\,\frac{1}{\Box^k} \;=\; \frac{\Box'}{\Box^k} - b\,\frac{1}{\Box^k}$$

(substitute $u = \Box = x^2+bx+c$ to integrate the next to last term). So that leaves (b), i.e. integrating $1/(x^2+bx+c)^k$. This is accomplished in two steps:

- Complete the square in the denominator: $x^2+bx+c = (x+b/2)^2 + (c-b^2/4)$. (Advice: *learn this technique*!) Substituting $u = x+b/2$ and $a = \sqrt{c-b^2/4}$ yields the integral

$$\int \frac{du}{(u^2 \pm a^2)^k}$$

which, after trig substitution, succumbs (painfully). It is worth memorizing the final formulas for the case $k=1$:

$$\boxed{\int \frac{du}{u^2+a^2} \;=\; \frac{1}{a}\tan^{-1}(\frac{u}{a})} \qquad \text{and} \qquad \boxed{\int \frac{du}{u^2-a^2} \;=\; \frac{1}{2a}\ln\left|\frac{u-a}{u+a}\right|}$$

These can be verified by differentiating the right hand sides, or derived by integrating using the substitution $u = a\tan\theta$ in the first and $u = a\sec\theta$ in the second.

† We assume all the our polynomials $a_0 + a_1x + a_2x^2 + \cdots$ are real, meaning the $a_i$'s are all real numbers.

## Partial Fraction Decompositions (PFDs)

### Examples

(1) $$\frac{1}{x^2-4} \;=\; \frac{1}{(x-2)(x+2)} \;=\; \frac{a}{x-2}+\frac{b}{x+2}$$

To find $a$ and $b$, add the fractions on the right to get $\dfrac{1}{x^2-4} \;=\; \dfrac{a(x+2)+b(x-2)}{x^2-4}$

Equate the numerators, $a(x+2)+b(x-2)=1$, and then the coefficients of $x$ and 1:

$$\begin{cases} a+b &= 0 \\ 2a-2b &= 1 \end{cases} \rightsquigarrow \; a=-b=1/4$$

Therefore $$\frac{1}{x^2-4} \;=\; \frac{1}{4}\left(\frac{1}{x-2}-\frac{1}{x+2}\right) \quad \left(\rightsquigarrow \int \frac{dx}{x^2-4} \;=\; \frac{1}{4}\ln\left|\frac{x-2}{x+2}\right|\right)^{\dagger}$$

(2) $$\frac{x}{x^2-2x+2} \;=\; \frac{x}{(x-2)(x-1)} \;=\; \frac{a}{x-2}+\frac{b}{x-1} \;=\; \cdots \;=\; \frac{2}{x-2}-\frac{1}{x-1}$$

**Short cut : the Heaviside method** If $(x-a)^n$ is the largest power of $x-a$ in the denominator of a rational function $f$, then the coefficient of $1/(x-a)^n$ in the PFD can be obtained by plugging $a$ into $f(x)/(x-a)^n$:

$$f(x) \;=\; \frac{p(x)}{(x-a)^n q(x)} \;=\; \frac{p(a)/q(a)}{(x-a)^n}+\cdots \qquad \text{(where } p(a) \text{ and } q(a) \text{ are both nonzero)}$$

Why? Multiply by $(x-a)^n$ and then take the limit as $x \to a$. This gives a quick way of redoing the two examples above. For example in (2), $a = x/(x-1)|_{x=2} = 2/1 = 2$

### More Examples

(3) $$\frac{x+1}{x^3-2x^2+x} \;=\; \frac{x+1}{x(x-1)^2} \;=\; \frac{a}{x}+\frac{b}{x-1}+\frac{c}{(x-1)^2} \;=\; \cdots \;=\; \frac{1}{x}-\frac{1}{x-1}+\frac{2}{(x-1)^2}$$

Note that Heaviside readily gives $a$ and $c$, but not $b$.

(4) Here's an example with an irreducible quadratic factor in the denominator:

$$\frac{3}{x^3+1} \;=\; \frac{3}{(x+1)(x^2-x+1)} \;=\; \frac{a}{x+1}+\frac{bx+c}{x^2-x+1} \;=\; \cdots \;=\; \frac{1}{x+1}+\frac{6-3x}{x^2-x+1}$$

Note that Heaviside only gives us $a$; we find $b$ and $c$ by solving a system of equations.

Once we have computed these PFDs, we can (for example using (4)) compute

$$\int \frac{3}{x^3+1}\,dx \;=\; \ln|x+1| - \frac{1}{2}\ln|x^2-x-1| + \sqrt{3}\tan^{-1}\frac{2x-1}{\sqrt{3}}.$$

The details are left to the reader!

† Try using this technique to give an alternative derivation of the last formula on page 20.

## 5.8 L'Hôpital's Rule (for evaluating limits of quotients)

Recall that $\lim_{x\to a} f(x)/g(x)$ can be computed as $\lim_{x\to a} f(x)/\lim_{x\to a} g(x)$ provided the limit in the denominator is nonzero. If it is zero but the limit in the numerator is nonzero, then the original limit is $\pm\infty$ (assuming that $f$ is continuous and nonzero near $a$).

But what if both $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ are zero (the "0/0 case"), or both are infinite (the "$\infty/\infty$" case). This is when L'Hôpital's Rule applies, asserting that

$$\boxed{\lim_{x\to a}\frac{f(x)}{g(x)} \;=\; \lim_{x\to a}\frac{f'(x)}{g'(x)}}$$

provided $f$ and $g$ are differentiable and the right hand limit exists.†

**Remarks**

① L'Hôpital's Rule applies when $a = \pm\infty$ (analyzing the "asymptotic behavior" of $f/g$)

② One must always be in either a 0/0 or $\infty/\infty$ case to apply the rule. For example

$$\lim_{x\to 1}\frac{x^2}{x} \;\neq\; \lim_{x\to 1}\frac{2x}{1} \;=\; 2$$

since we're in a "1/1 case". In fact the left hand limit is 1 (always first try just plugging in when investigating a limit).

③ The rule can also be used to analyze limits of products $f(x)h(x)$ in the "$0\cdot\infty$" case: Write $f\cdot h$ as $f/(1/h)$ or $h/(1/f)$, and apply the 0/0 or $\infty/\infty$ case of L'Hôpital's Rule.

**Examples**

① $\displaystyle \lim_{x\to 0}\frac{\sin x}{x} \underset{0/0}{=} \lim_{x\to 0}\frac{\cos x}{1} \;=\; 1$ (This is actually a circular reasoning, since we needed this limit to show $\sin' = \cos$)

② $\displaystyle \lim_{x\to 1}\frac{x^3-1}{x-1} \underset{0/0}{=} \lim_{x\to 1}\frac{3x^2}{1} = 3$ (This can also be computed by factoring the numerator $x^3-1=(x-1)(x^2+x+1)$, cancelling the $(x-1)$'s, and then plugging in $x=1$)

③ $\displaystyle \lim_{x\to\infty}\frac{x}{e^x} \underset{\infty/\infty}{=} \lim_{x\to\infty}\frac{1}{e^x} \;=\; 0$ (the last limit is a "$1/\infty$" case)

④ $\displaystyle \lim_{x\to\infty}\frac{x^2}{e^x} \underset{\infty/\infty}{=} \lim_{x\to\infty}\frac{2x}{e^x} \underset{\infty/\infty}{=} \lim_{x\to\infty}\frac{2}{e^x} \;=\; 0$ (the rule is applied twice)

⑤ $\displaystyle \lim_{x\to 0}\frac{1-\cos x}{x\sin x} \underset{0/0}{=} \lim_{x\to 0}\frac{\sin x}{\sin x + x\cos x} \underset{0/0}{=} \lim_{x\to 0}\frac{\cos x}{2\cos x - \sin x} \;=\; \frac{1}{2}$

⑥ $\displaystyle \lim_{x\to 0} x\ln x \underset{0\cdot\infty}{=} \lim_{x\to 0}\frac{\ln x}{1/x} \underset{0/0}{=} \lim_{x\to 0}\frac{1/x}{-1/x^2} \;=\; \lim_{x\to 0} -x \;=\; 0$

† The proof of L'Hôpital's Rule (in the 0/0 case, assuming $f'$ and $g'$ are continuous) goes roughly as follows: We can assume $f(a) = g(a) = 0$ without affecting the limits. Then

$$\lim_{x\to a}\frac{f(x)}{g(x)} \;=\; \lim_{x\to a}\frac{\dfrac{f(x)-f(a)}{x-a}}{\dfrac{g(x)-g(a)}{x-a}} \;=\; \frac{\displaystyle\lim_{x\to a}\frac{f(x)-f(a)}{x-a}}{\displaystyle\lim_{x\to a}\frac{g(x)-g(a)}{x-a}} \;=\; \frac{f'(a)}{g'(a)} \;=\; \lim_{z\to a}\frac{f'(x)}{g'(x)}$$

## 6.6 Improper Integrals

These are definite integrals

$$\int_a^b f(x)\,dx$$

for which one or both of the bounds $a, b$ are infinite, and/or for which $f$ is 'bad' at finitely many points in $[a, b]$ (meaning $f(x)$ is unbounded as $x$ approaches those points). Here, we only consider the following three cases:

- One infinite bound Then define

$$\int_a^\infty f(x)\,dx \;=\; \lim_{t\to\infty}\int_a^t f(x)\,dx \quad \text{and} \quad \int_{-\infty}^b f(x)\,dx \;=\; \lim_{t\to-\infty}\int_t^b f(x)\,dx$$

If the limit exists, we say that the improper integral converges, and otherwise it diverges.

- Both bounds are infinite Then define

$$\int_{-\infty}^\infty f(x)\,dx \;=\; \int_{-\infty}^0 f(x)\,dx + \int_0^\infty f(x)\,dx\ ^\dagger$$

provided both improper integrals on the right converge, in which case we say that the original integral converges; otherwise it diverges.

- Finite bounds, but one of them is 'bad' (meaning $f(x)$ is unbounded as $x$ approaches the bad endpoint from within the interval $[a, b]$). For example, if $b$ is bad, then define

$$\int_a^b f(x)\,dx \;=\; \lim_{t\uparrow b}\int_a^t f(x)\,dx$$

**Examples** (sketch the integrands to interpret the results in terms of areas)

① $\displaystyle\int_1^\infty \frac{dx}{x} \;=\; \lim_{t\to\infty}(\ln x\big|_1^t) \;=\; \lim_{t\to\infty}\ln t \;=\; \infty$ (abbreviated $\displaystyle\int_1^\infty \frac{dx}{x} \;=\; \ln x\,\big|_1^\infty \;=\; \infty$)

so the integral diverges, whereas $\displaystyle\int_1^\infty \frac{dx}{x^2}$ converges to $(-1/x)\big|_1^\infty \;=\; 0-(-1) \;=\; 1$

② $\displaystyle\int_0^\infty \frac{dx}{x^2+1} \;=\; \tan^{-1}(x)\Big|_0^\infty \;=\; \frac{\pi}{2}-0 \;=\; \frac{\pi}{2}$ ; it follows that

$$\int_{-\infty}^\infty \frac{dx}{x^2+1} \;=\; \int_{-\infty}^0 \frac{dx}{x^2+1}+\int_0^\infty \frac{dx}{x^2+1} \;=\; 2\int_0^\infty \frac{dx}{x^2+1}\,dx \;=\; \frac{\pi}{2}+\frac{\pi}{2} \;=\; \pi$$

③ $\displaystyle\int_{-\infty}^\infty 2xe^{1-x^2}\,dx \;=\; (-e^{1-x^2})\big|_{-\infty}^0+(-e^{1-x^2})\big|_0^\infty \;=\; (-e-0)+(0-(-e)) \;=\; 0$

④ $\displaystyle\int_0^1 \frac{dx}{x^2-1} \;=\; \ln\left|\frac{x-1}{x+1}\right|\Big|_0^1 \;=\; \ln 0-\ln 1 \;=\; -\infty$, so the integral diverges

**More generally** both endpoints may be 'bad', e.g. $\displaystyle\int_{-1}^1 \frac{dx}{x^2-1}$, or we may have internal 'bad' points, e.g. $\displaystyle\int_0^2 \frac{dx}{x^2-1}$ at $x=1$. Note: both of these integrals diverge.

† Note: You can replace 0 by any number $a$ in the definition, i.e. $\int_{-\infty}^\infty f = \int_{-\infty}^a f + \int_a^\infty f$

## 7.2–7.3 Computing Volumes

### Solids of Revolution

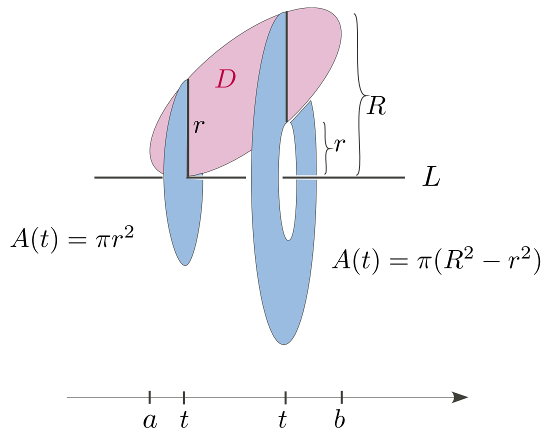
Consider a solid $S$ formed by rotating a domain $D$ in the plane about a line $L$ that meets the domain – if at all – along a portion of its boundary. It may be difficult to draw the entire solid $S$, but not so hard to sketch its intersections with planes perpendicular to $L$, called slices of $S$, or with cylinders centered around $L$, called shells. We would like to express the volume $V$ of $S$ as an integral in a suitable variable $t$.

Method of Slices

Choose the $t$-axis parallel to $L$. If $a$ and $b$ are the bounds of $D$, then

$$V \;=\; \int_a^b A(t)\,dt \qquad \text{(a slice integral)}$$

where $A(t)$ is the area of the slice at level $t$. These slices will either be disks where $R$ meets $L$, or annuli (or unions of annuli) where $R$ is disjoint from $L$, as shown below. Thus $A(t)$ will typically be of the form $\pi r^2$ or $\pi(R^2 - r^2)$, where $r$ and $R$ must be expressed *in terms of* $t$ (often a sketch will help in accomplishing this).




### Examples

1 Let $D$ be bounded by the $x$-axis, the line $x = 2$ and the curve $y = x^2$. Find the volume of the solid obtained by rotating $D$ about

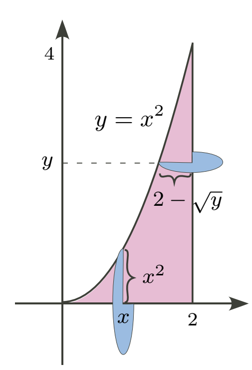
a) the $x$-axis: $V \;=\; \displaystyle\int_0^2 \pi(x^2)^2\,dx \;=\; (\pi x^5/5)\big|_0^2 \;=\; 32\pi/5$

b) the line $y = -1$: $V = \displaystyle\int_0^2 \pi((1+x^2)^2 - 1^2)dx$

$\;=\; \int_0^2 \pi(2x^2 + x^4)\,dx \;=\; \pi(16/3 + 32/5) \;=\; 176\pi/15$

c) the $y$-axis (note that the curve can be written as $x = \sqrt{y}$):

$$V \;=\; \int_0^4 \pi(2^2 - \sqrt{y}^{\,2})\,dy \;=\; \pi(4y - y^2/2)\big|_0^4 \;=\; 8\pi$$




(sketches for parts a & d)

d) the line $x = 2$: $V \;=\; \displaystyle\int_0^4 \pi(2 - \sqrt{y})^2\,dy \;=\; \int_0^4 \pi(4 - 4\sqrt{y} + y)\,dy \;=\; 8\pi/3$

② Set up slice integrals (but do not evaluate) to compute the volumes of solids obtained by rotating the domains bounded by

a) the $x$-axis and the graph of $y = \cos x$ for $x$ in $[0, \pi/2]$, about both axes

$$x\text{-axis: } \int_0^{\pi/2} (\cos x)^2\, dx \qquad\qquad y\text{-axis: } \int_0^1 \pi(\cos^{-1} x)^2\, dy$$

b) $y = 2x$ and $y = x^2$, about both axes

$$x\text{-axis: } \int_0^2 \pi(x^2 - x^4)\, dx \qquad\qquad y\text{-axis: } \int_0^1 \pi(y - y^2)\, dy$$

c) the $x$-axis, $x = h$ and $y = rx/h$, about the $x$-axis (this is an ice cream cone)

$$\int_0^h \pi \left(\frac{rx}{h}\right)^2 dx \qquad \left(= \frac{\pi r^2 h}{3}\right)$$

d) the circle of radius 1 about the point $(2, 0)$, about the $y$-axis (this is a donut)

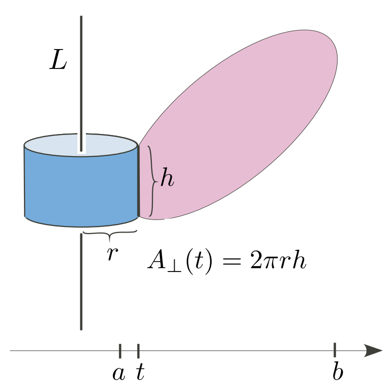
$$\int_{-1}^1 \pi \left((2 + \sqrt{1 - y^2})^2 - (2 - \sqrt{1 - y^2})^2\right) dy \;=\; 8\pi \int_{-1}^1 \sqrt{1 - y^2}\, dy \qquad (= 8\pi^2)$$

$\boxed{\text{Method of Shells}}$

Choose the $t$-axis perpendicular to $L$. If $a$ and $b$ are the bounds of $R$, then

$$V_\perp \;=\; \int_a^b A_\perp(t)\, dt \qquad \text{(a shell integral)}$$

where $A_\perp(t)$ is the area of the cylinder at level $t$, as shown below, and so will typically be of the form $2\pi r h$ where $r$ and $y$ must be computed *in terms of* $t$ as before:




**Examples**

① Repeat the problems in example ① above, but now using shells. The integrals are:

a) $V_\perp = \int_0^4 2\pi y(2 - \sqrt{y})\, dy = 32\pi/5$ b) $V_\perp = \int_0^4 2\pi(1 + y)(2 - \sqrt{y})\, dy = 176\pi/15$

c) $V_\perp = \int_0^2 2\pi x \cdot x^2\, dx = 8\pi$ d) $V_\perp = \int_0^2 2\pi(2 - x)x^2\, dx = 8\pi/3$

② Should slices or shells be used to compute the volume of the solid obtained by rotating the domain bounded by $y = 2x - x^2$, and the lines $x = 0$, $x = 2$ and $y = 3$, about the $x$-axis? (Answer: slices) What about the $y$-axis? (Answer: shells) Do you see why?

## 7.4, 9.2 Computing Arc Lengths and Surface Areas

Motivating discussion under construction ...

Key observation: the 'infinitessimal' Pythogorean theorem shows that the 'element' $ds$ of arc length is

$$ds = \sqrt{dx^2 + dy^2}.$$

For the graph of a function $y = f(x)$ we would like to express this in terms of $x$, and so we multiply and divide by $dx$ :

$$ds = \sqrt{\frac{dx^2 + dy^2}{dx^2}}\ dx = \sqrt{1 + f'(x)^2}\ dx.$$

For a parametrized curve $x = x(t), y = y(t)$, we multiply and divide by $dt$ :

$$ds = \sqrt{\frac{dx^2 + dy^2}{dt^2}}\ dt = \sqrt{x'(t)^2 + y'(t)^2}\ dt.$$

This leads to a formula for the length $L = \displaystyle\int_0^L ds$ of the graph of $y = f(x)$ for $a \le x \le b$ :

$$L = \int_a^b \sqrt{1 + f'(x)^2}\ dx$$

and the length of the parametrized curve $x = x(t), y = y(t)$ for $t_0 \le t \le t_1$ :

$$L = \int_{t_0}^{t_1} \sqrt{x'(t)^2 + y'(t)^2}\ dt.$$

**Examples** ...

## 7.5 Computing Surface Areas

Building on the arc length formulas, one can derive formulas for the areas of surfaces of revolution. For example, the area $A$ of the surface obtained by rotating the graph of $y = f(x)$ for $a \le x \le b$ about the $x$-axis is given by

$$A = \int_a^b 2\pi f(x) \sqrt{1 + f'(x)^2}\ dx$$

since each $x$ in $[a, b]$ corresponds to a circle on the surface of length $2\pi r = 2\pi f(x)$ (because we are rotating about the $x$-axis), and when it is thickened infinitessimally to $dx$, this circle becomes a infinitessimal 'slanted cylinder' of width $ds = \sqrt{1 + f'(x)^2}\, dx$. If we were rotating around the line $y = -1$ instead, then $r$ would become $f(x) + 1$ while $ds$ would remain the same, so the area would be the integral of $2\pi(f(x) + 1)\sqrt{1 + f'(x)^2}$.

Similar formulas can be written down for parametric curves. The principle is always the same. The area is given by an integral of the form

$$A = \int_*^* 2\pi r ds$$

where $r$ and $ds$ are expressed in terms of the appropriate variable.

**Examples** ...

## 7.7 Differential Equations

**Definition** An ordinary differential equation, abbreviated ODE, is an equation in an independent variable $x$, a dependent variable $y$, and the derivatives $y', y'', \ldots$ of $y$ with respect to $x$. Thus a general ODE can be written in the form

$$F(x, y, y', y'', \ldots) \;=\; 0$$

for some function $F$. The highest order of a derivative that appears is called the order of the ODE. We will limit our discussion to first order equations $F(x, y, y') = 0$ that that can be rewritten in the special form

$$y' \;=\; f(x, y). \tag{1}$$

(excluding equations such as $y' + \sin(y') = xy$). A solution to (1) is a function $y = y(x)$ that satisfies (1), i.e. $y'(x) = f(x, y(x))$.

**Examples** ...

Equation (1) is called separable if $f(x, y)$ is of the form $h(x)/g(x)$, so it becomes

$$g(y)\, y' \;=\; h(x). \tag{2}$$

To solve this we go through the following formal procedure: Substitute $dy/dx$ for $y'$, then multiply by $dx$ on both sides ... giving $g(y)dy = h(x)dx$ ... and finally integrate both sides: $\int g(y)\, dy = \int h(x)\, dx$. Thus the solutions to (2) are exactly the solutions to

$$G(y) \;=\; H(x) + C \tag{3}$$

where $G$ and $H$ are any chosen antiderivatives of $g$ and $h$.[†] Do not forget to include the $+\,C$; it is essential in what follows. To complete the solution to (2), one then tries to rewrite (3) so as to solve for $y$ explicitly in terms of $x$.

**Examples** ...

**Remark** Not all first order ODEs are separable, in fact 'most' aren't. Furthermore there is no general method known for solving first order ODEs – even the ones in the special form (1) – although solutions always exist in theory – see below. There are, however, some standard techniques for attacking non-separable equations, e.g. the method of integrating factors. These are treated in any introductory course on differential equations.

In applications, one often imposes an initial condition $y = y_0$ when $x = 0$ (or more generally $y = y_0$ when $x = x_0$), i.e. one seeks a solution $Y = f(x)$ for which $f(0) = y_0$. The resulting set up is called a first order initial value problem (IVP for short) :

$$\begin{cases} y' \;=\; f(x, y) \\ y \;=\; y_0 \;\; \text{when} \;\; x \;=\; x_0 \end{cases} \tag{4}$$

The initial condition allows us to solve for the unknown constant $C$, and thus to pin down a unique solution.

**Theorem** Existence and Uniqueness Theorem (no proof given here) *Under mild conditions on the function $f$* (which we don't specify here)*, the* IVP *in* (4) *has a unique solution.*

[†] This is rigorously proved as follows: $y$ satisfies (3) $\iff G(y) = H(x) + C \iff G'(y)y' = H'(x)$ (since functions have the same derivative if and only if they differ by a constant, by the Mean Value Theorem) $\iff g(y)y' = h(x) \iff y$ satisfies (2).

**Example** (1) Solve the IVP

$$\begin{cases} y' = 2xy^2 \\ y = 1 \text{ when } x = 0\,. \end{cases}$$

Separating variables we arrive at

$$\int \frac{1}{y^2}\,dy \;=\; \int 2x\,dx \quad \text{which implies} \quad \frac{-1}{y} \;=\; x^2 + C.$$

Substituting in the initial condition gives $-1 = 0 + C$, so $C = -1$. Thus

$$y \;=\; \frac{-1}{x^2-1} \;=\; \frac{1}{1-x^2}.$$

(2) (The logistic equation) Consider a population $P(t)$ constrained (by the environment or some other factor) to a maximum carrying capacity of $M$. This is modeled by the IVP

$$\begin{cases} P \;=\; kP(M-P) \\ P \;=\; P_0 \text{ when } t = 0\,. \end{cases}$$

Separating variables, ...

## 8.1 Sequences

A sequence is an ordered, infinite list of numbers: $s_1, s_2, s_3, \ldots$ indexed by the natural numbers, or more formally, a function $s : \mathbb{N} \to \mathbb{R}$, where $s_n = s(n)$. We say that $s_n$ converges to a number $s$, or that $s$ is the limit of the sequence $s_n$, written

$$\lim_{n\to\infty} s_n \;=\; s \text{ , or simply } s_n \longrightarrow s$$

if we can make $s_n$ as close to $s$ as we want by choosing $n$ sufficiently large. More precisely, this means that for every $\varepsilon > 0$, there is an $N$ such that $|s_n - s| < \varepsilon$ for all $n > N$.

If $s_n$ does not converge, we say that it diverges, which can happen in a variety of ways. It can 'diverge to $\infty$' or 'to $-\infty$', or 'by oscillation', or in more unpredictable ways.

**Examples** (1) $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots, \frac{1}{n}, \ldots$ : $s_n$ converges to 0 (also written $s_n \longrightarrow 0$)

(2) $-1, 1, -1, 1, \ldots, (-1)^n, \ldots$ : $s_n$ diverges (by oscillation)

(3) $1, 4, 9, 16, \ldots, n^2, \ldots$ : $s_n$ diverges to $\infty$ (also written $s_n \longrightarrow \infty$, by abuse of notation)

(4) The prime sequence : $2, 3, 5, 7, 11, 13, \ldots, p_n, \ldots \longrightarrow \infty$ (a formula for $p_n$ is unknown)

(5) The Fibonacci sequence : $1, 1, 2, 3, 5, 8, \ldots, F_n, \ldots \longrightarrow \infty$ ; Do you see the pattern? Each term is the sum of the previous two. There is a famous formula (called Binet's formula) for the $n$th Fibonacci number:

$$F_n \;=\; \frac{r_+^n - r_-^n}{\sqrt{5}} \quad \text{where} \quad r_\pm = \frac{1 \pm \sqrt{5}}{2}.$$

The number $r_+$ is called the golden ratio.

Just as for limits of functions, limits of sequences obey the:

**Limit laws** If $a_n \longrightarrow a$ and $b_n \longrightarrow b$, then

- $a_n \pm b_n \longrightarrow a \pm b$
- $a_n b_n \longrightarrow ab$
- $a_n / b_n \longrightarrow a/b$ provided $b \neq 0$

**Remark** If $f : \mathbb{R} \to \mathbb{R}$ is a function with $f(n) = s_n$ and $\lim_{x\to\infty} f(x) = s$, then $s_n \longrightarrow s$. Thus we can sometimes use calculus to compute limits of sequences. For example

$$\frac{4n^2+5n-1}{2n^2+9} \longrightarrow 2 \quad \text{since} \quad \lim_{x\to\infty} \frac{4n^2+5n-1}{2n^2+9} \underset{\text{L'Hop}}{=} \lim_{x\to\infty} \frac{8n+5}{4n} \underset{\text{L'Hop}}{=} \lim_{x\to\infty} \frac{8}{4} = 2$$

This limit can also be computed by the trick of dividing numerator and denominator by the largest power of $n$ that appears, in this case $n^2$. This yields the expression:

$$\frac{4+5/n-1/n^2}{2+9/n^2}$$

Since all the terms involving $n$ go to zero, it follows from the basic fact that the limit of a below that the limit is $4/2 = 2$

DID NOT TEX UP NOTES FROM THE REST OF THE SEMESTER ...