INFINITE FAMILIES OF HOMOLOGOUS 2-SPHERES IN 4-MANIFOLDS

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Abstract. It is shown that primitive ordinary homology classes in a simply-connected 4-manifolds are often represented by *infinitely* many topologically isotopic, embedded 2-spheres that are not smoothly isotopic, but that become smoothly isotopic after a single external stabilization, or after summing with an unknotted surface of genus greater than half the square of the class.

1. Introduction

Embedded surfaces, especially 2-spheres, have been an essential part of the study of smooth simply-connected 4-manifolds from the very beginning of the theory. Using Rohlin's theorem [44], Kervaire-Milnor [24] showed that not all 2-dimensional homology classes are represented by embedded spheres. Shortly after, Wall [49] established a general existence result, showing that all *primitive ordinary* classes in stabilized indefinite 4-manifolds are represented by spheres. Using the Atiyah-Singer G-signature theorem [2], Rohlin [45] and Hsiang-Szczarba [23] provided further restrictions on classes that could be represented by spheres, or even surfaces of higher genus, and gauge theoretic techniques (when b_2^+ is odd) gave definitive genus bounds [27, 39, 40] in many cases. A good overview of the existence questions can be found in [28, 19] and [31].

This paper addresses the uniqueness question for embedded spheres. We work throughout in the smooth category; existence and uniqueness questions in the *topological* category were treated by Lee and Wilczyiński in [29, 30]. To state our results, first recall some standard terminology. A 2-dimensional homology class α in a closed, simply-connected 4-manifold is *primitive* if it is *not* a non-trivial multiple of another class, and is of *divisibility* $d \geqslant 0$ if it is d times a primitive class. The *square* of α is its self-intersection number, denoted α^2. The class is *characteristic* if it is dual mod 2 to the second Stiefel-Whitney class of X, and is otherwise *ordinary*. Following Wall [48], α is of *type* 0 or 1 according to whether its associated primitive class is ordinary or characteristic. Note that all nonzero classes are of type 0 when X is even and of type 1 when X is odd with $b_2(X) = 1$; otherwise X has classes of both types, of any given divisibility.

A closed oriented surface F embedded in X is *simple*, following [29, 30], if $\pi_1(X - F)$ is abelian, hence cyclic of order the divisibility of $[F] \in H_2(X)$. An *external* or *internal stabilization* of (X, F) is the operation transforming (X, F) respectively to $(X, F) \# (S^2 \times S^2, \varnothing)$ or to $(X, F) \# (S^4, T^2)$ (where T^2 is an unknotted torus in S^4), while an *external blowup* transforms it to $(X, F) \# (\overline{\mathbb{C}P}^2, \varnothing)$. Two surfaces E and F in X are *stably distinct* if no diffeomorphism of X, even after arbitrarily many external blowups, carries E to F.

Theorem A. *There exist infinitely many closed simply-connected 4-manifolds X for which each primitive ordinary class α in $H_2(X)$ is represented by infinitely many simple, stably distinct 2-spheres that*

1) *become isotopic after one external stabilization,*

2) *become isotopic after at most $\lfloor \alpha^2/2 \rfloor + 1$ internal stabilizations, and*

3) *intersect pairwise in at most $5(\lfloor \alpha^2/2 \rfloor + 1)^2$ points,*

The 4-manifolds exhibiting this property include all the reduced manifolds $p\,\mathbb{C}P^2 \# q\,\overline{\mathbb{C}P}^2$ for even $p \geqslant 4$, $q \geqslant 5p+2$, and for odd p congruent to 3 mod 4 with $p \geqslant 7$, $q \geqslant 5p+6$ (see also Remark 4.3). *The analogous result holds for classes of square $n \leqslant 0$ in $-X$.*

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Remarks 1.1. a) The spheres in Theorem A representing any given homology class are pairwise *topologically* isotopic. Indeed this is true for any pair of homologous, simple, primitive 2-spheres in an indefinite simply-connected 4-manifold X, and when X is strongly indefinite, for spheres of arbitrary divisibility that are of square 0 and type 0 [29, 30, 20] (cf. Remark 3.6). Furthermore, it is known that any pair of topologically isotopic 2-spheres in X become smoothly isotopic after some number e of external stabilizations [43, Theorem 1.4], and also after some number i of internal stabilizations [7, Theorem 1]. It is also evident that such spheres must intersect in at least $|n|$ points, where n is the square of the associated homology class, but they may have a larger *minimal geometric intersection number* m (see [17, 22, 46] for some earlier work on such 'excess intersections'). So the real content of the theorem is the construction of *infinite* families of topologically isotopic but smoothly distinct spheres for which $e = 1$, and i and m are uniformly bounded above by, respectively, a linear and quadratic function of the self-intersection number n.

b) We do not know whether these bounds on i and m (defined in the preceding remark) can be improved. But with regard to the external stabilization number e, the authors and H. Schwartz have recently shown (building on Gabai's work [16] on the 4-dimensional 'light bulb' theorem) that *one external stabilization is enough in general*: Any pair of simple spheres representing a primitive ordinary homology class in a 4-manifold become smoothly isotopic after summing with a *single* $S^2 \times S^2$ [4]. This provides an alternative proof to the one given here, which predated it, of condition **1)** , and considerably widens the class of 4-manifolds exhibiting this phenomena.

c) Examples of infinite families of homologous but smoothly distinct simple embedded 2-spheres have been known for many years, following immediately from the foundational work of Wall [49] and Donaldson [10][11]. The first such examples, however, satisfying conditions **1)** and **3)** in the theorem (or any uniform "external stabilization" or "excess intersection" bounds for that matter) were constructed by the authors in [3], and later shown by Baykur and Sunukjian [7] to satisfy **2)** . These were spheres of square +1 in $p\,\mathbb{C}P^2 \# q\,\overline{\mathbb{C}P}^2$ for even $p \geqslant 4$ with $q \geqslant 5p$, a slightly broader class of 4-manifolds than in Theorem A when p is even. By Wall's work on quadratic forms [48] and diffeomorphisms of 4-manifolds [49], it follows that *all ordinary classes of square* +1 in these 4-manifolds satisfy the conclusions of Theorem A; note that every class of square +1 in a simply-connected 4-manifold is primitive, and any sphere S representing such a class is simple since the meridian of S contracts in the boundary 3-sphere of a tubular neighborhood of S.

As an added feature it will be seen that any pair of spheres in such a family must intersect in *more than one point*, and more generally any pair of spheres in the families constructed in this paper representing classes of *nonzero* square n will have minimal geometric intersection number strictly greater than $|n|$ (see Proposition 3.3).

There is a similar stabilization result for spheres representing non-primitive homology classes of square 0 (see also Remark 4.3b regarding classes of square ± 4).

Theorem B. *There exist infinitely many closed simply-connected* 4*-manifolds* X (*including those listed in Theorem A, as well as those with one fewer* $\overline{\mathbb{C}P}^2$*– summand*) *for which each nontrivial class in* $H_2(X)$ *of square* 0 *and type* 0 *is represented by infinitely many simple, stably distinct* 2*-spheres that become isotopic after one external or d internal stabilizations, where d is the divisibility of the class, and that pairwise intersect in at most* $5d^2$ *points.*

Remark 1.2. Infinite families of smoothly non-isotopic (but topologically isotopic) surfaces of higher genus have been known for some years. The rim surgery technique of Fintushel and Stern [15] (see also [47]) gives examples for surfaces in primitive homology classes, while surfaces in non-primitive classes were constructed by the annulus surgery technique of Finashin [34] and the twisted rim surgery of Kim [35, 36].

These results suggest the following problems:

Problem 1.3. Is there a 4-manifold for which *every* primitive ordinary homology class (of arbitrary square) satisfies the conclusion of Theorem A, or one for which *at least one* primitive ordinary class does not satisfy that conclusion.

Problem 1.4. Is it possible that any smoothly embedded surface of minimal genus in its homology class in a 4-manifold is topologically isotopic to infinitely many smoothly distinct surfaces?

2. Preliminaries

Isotopy of surfaces in simply-connected 4-manifolds.

It is well known that any isotopy of a smooth simply-connected n-manifold X that moves a point p around a loop can be replaced by an isotopy that fixes p. More precisely, the inclusion $\mathrm{Diff}_p(X) \subset \mathrm{Diff}(X)$ (where $\mathrm{Diff}(X)$ is the group of orientation preserving diffeomorphisms of X, and $\mathrm{Diff}_p(X)$ is the subgroup of diffeomorphisms that fix $p \in X$) induces an isomorphism

$$\pi_0 \,\mathrm{Diff}_p(X) \;\cong\; \pi_0 \,\mathrm{Diff}(X)$$

of the corresponding mapping class groups. To prove this, recall that "evaluation at p" defines a bundle projection $\mathrm{Diff}(X) \to X$ with fiber $\mathrm{Diff}_p(X)$ (see e.g. [33, §4.2.3] for an elementary proof), and so the result follows from the homotopy sequence of this fibration.

This result fails in general if p is replaced by an n-ball, but is true up to Dehn twists along the boundary of the ball (cf. [3, Proposition 5.2]). This leads to a relative version of the above result: Any isotopy of a submanifold $F \subset X$ of codimension $\geqslant 2$ that carries a disk $D \subset F$ back to itself can be replaced by an isotopy that fixes D. Here is a precise statement for the case needed here of surfaces in 4-manifolds, with a short direct proof. We use the notation $\mathrm{Emb}(M, N)$ for the space of smooth embeddings of one manifold M in another N (with the C^∞ topology) and $\mathrm{Emb}_D(M, N)$ for the subspace of embeddings that fix $D \subset M$.

Lemma 2.1. *Let F be a surface embedded in a simply-connected 4-manifold X, and (B, D) be a standard* (4-ball, 2-disk) *pair in X with $B \cap F = D$. Then the inclusion* $\mathrm{Emb}_D(F, X) \subset \mathrm{Emb}(F, X)$ *induces an isomorphism*

$$\pi_0 \,\mathrm{Emb}_D(F, X) \;\cong\; \pi_0 \,\mathrm{Emb}(F, X).$$

It follows that if two embeddings of F in X agree on D and are freely isotopic, then they are ambiently isotopic fixing B.

Proof. By a theorem of Palais [42], $\mathrm{Emb}(F, X)$ fibers over $\mathrm{Emb}(D, X)$ with fiber $\mathrm{Emb}_D(F, X)$, and $\mathrm{Emb}(D, X)$ in turn fibers over $\mathrm{Emb}(p, X) = X$ (for any $p \in D$) with fiber $\mathrm{Emb}_p(D, X)$. Noting that $\mathrm{Emb}_p(D, X)$ is homotopy equivalent to the Stiefel manifold $V_2(\mathbb{R}^4)$ of 2-frames in 4-space, and that $\pi_1 V_2(\mathbb{R}^4)$ is trivial, the homotopy sequence of the second fibration shows that $\pi_1 \,\mathrm{Emb}(D, X)$ is trivial. The first assertion in the lemma now follows from the homotopy sequence of the first fibration, since $\pi_0 \,\mathrm{Emb}(D, X)$ is trivial by Palais' disk theorem [41], and the last statement follows from the relative isotopy extension theorem [21]. □

Remark 2.2. A useful consequence of this lemma (not specifically needed here but presumably known to experts) is that *pairwise connected sums of surfaces in simply-connected 4-manifolds are well-defined up to isotopy, depending only on the isotopy classes of the original surfaces.* To show this, note that such sums can be constructed by first connected summing the ambient 4-manifolds along 4-balls disjoint from the surfaces, and then tubing the surfaces together with a tube that meets the separating 3-sphere in a single circle. From this perspective the claim is that the result is independent of the tube, which follows easily from the lemma.

Blowups, blowdowns, surgery and Gluck twists.

Let X be a 4-manifold. *Blowing up* a point $p \in X$ is the operation $X \rightsquigarrow X \# \overline{\mathbb{C}P}^2$, which replaces p by the 2-sphere $\overline{\mathbb{C}P}^1$ of square -1. If p lies on a surface $F \subset X$ of square n, then the blowup transforms F into the surface $F \# \overline{\mathbb{C}P}^1$ of square $n-1$. Blowups of the opposite orientation $X \rightsquigarrow X \# \mathbb{C}P^2$, referred to as *anti-blowups* or $+1$-*blowups*, are also allowed. Conversely, one can *blow down* any 2-sphere S of square ± 1 to give the 4-manifold X/S, so written because this operation collapses S to a point.

Now consider a 2-sphere $S \subset X$ of square 0. There are two familiar operations that can be performed on X along S:

- *Surgery* $X \rightsquigarrow X/S = (X - \mathrm{int}(S \times D^2)) \cup B^3 \times S^1$ (a rationale for this notation, the same as for blowdowns, is given in Definition 3.1 below). This replaces S by a circle.
- *Gluck twist* $X \rightsquigarrow \mathrm{Gluck}_X(S) = (X - \mathrm{int}(S \times D^2)) \cup_\tau S \times D^2$ where $\tau : S \times S^1 \to S \times S^1$ maps (s, θ) to $(\mathrm{rot}_\theta(s), \theta)$. This replaces S by another 2-sphere.

Surgery generally alters the algebraic topology of X. Gluck twists can alter the algebraic topology as well – e.g. twisting a fiber in the trivial bundle $S^2 \times S^2$ yields the twisted bundle $S^2 \tilde{\times} S^2$ – but it is not known whether twisting a *null-homologous* sphere in an oriented 4-manifold X can ever change the smooth topology of X. By restricting to null-homologous spheres, one has a natural isomorphism $H_2(X) \to H_2(\mathrm{Gluck}_X(S))$ preserving the intersection form, and in this case the twist at least does not change the gauge-theoretic invariants of the 4-manifold. We record a quick proof of this fact below, based on the blowup formula, for use in proving Proposition 3.3 below.

Lemma 2.3. (Folklore) *If S is a null-homologous 2-sphere embedded in a 4-manifold X, then the (Donaldson, Seiberg-Witten, or Bauer-Furuta) invariants of X and $\mathrm{Gluck}_X(S)$ agree.*

If $b_2^+(X) > 1$, then the gauge theoretic invariants in the lemma are well-defined, but to compare them on different manifolds requires an identification of the cohomology groups (to pick out basic classes or KO-orientations in the case of Seiberg-Witten or Bauer-Furuta invariants, or to compare polynomial invariants in the Yang-Mills setting). As remarked above, a null-homologous Gluck twist provides such an identification. This identification is also needed when $b_2^+(X) = 1$, in which case the invariants of X and $\mathrm{Gluck}_X(S)$ are to be computed in the corresponding chambers in their cohomology groups. Note that the presence of a sphere of square 0 representing a nontrivial homology class implies (when $b_2^+(X) > 1$) the vanishing of both Donaldson and Seiberg-Witten invariants, so one could trivially extend the lemma to cover that case.

Proof of Lemma 2.3. A Gluck twist along S has the same effect as blowing up a point on S and then blowing down the resulting sphere of square -1 (the proper transform of S). This is a standard argument in Kirby's calculus of framed links [25] (see Figure 1 below, or [37, Proposition 6.2]).

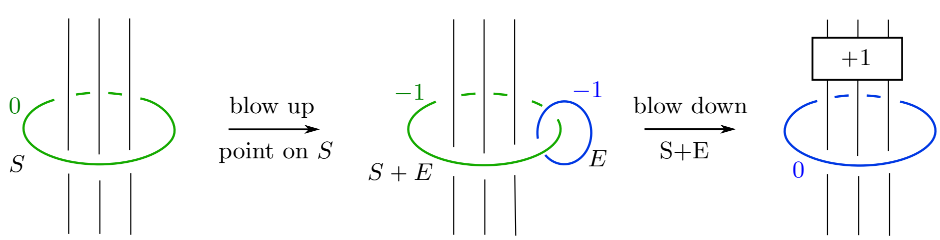

Figure 1. A Gluck twist as a blowup and blowdown

It follows that the gauge theoretic invariants of $\mathrm{Gluck}_X(S)$ can be calculated from those of X using the relevant blowup formulas [39, 12, 6]. For example, the Seiberg-Witten invariant for any characteristic $K \in H_2(X)$ (letting K also denote the corresponding class in $H_2(\mathrm{Gluck}_X(S))$) is

$$\begin{aligned}\mathrm{SW}_X(K) &= \mathrm{SW}_X(K+S) \qquad \text{(since } S \text{ is null-homologous)}\\ &= \mathrm{SW}_{X\#\overline{\mathbb{C}P}^2}(K+S+E) = \mathrm{SW}_{(X\#\overline{\mathbb{C}P}^2)/(S+E)}(K) = \mathrm{SW}_{\mathrm{Gluck}_X(S)}(K)\end{aligned}$$

where $E = \overline{\mathbb{C}P}^1 \subset \overline{\mathbb{C}P}^2$. The proof for Donaldson and Bauer-Furuta invariants is similar. □

Wall's transitivity results.

Theorems A and B assert that *every* homology class satisfying certain conditions is represented by infinitely many smoothly distinct spheres. Our construction actually finds spheres in one such class α, and then makes use of classic results of Wall to establish that all the classes satisfying the given conditions are related to α by diffeomorphisms. We recall Wall's results for the reader's convenience. Part a) is the 'Conclusion' on page 337 of [48], while part b) is Theorem 2 in [49].

Theorem 2.4. (Wall) a) *The automorphisms of any unimodular, strongly indefinite form (meaning the rank and absolute signature of the form differ by at least* 4*) act transitively on elements of given square, divisibility and type.*

b) *If X is a closed, simply-connected* 4*-manifold with an indefinite intersection form, then every automorphism of the intersection form of $X \# (S^2 \times S^2)$ is realized by a diffeomorphism.*

The manifolds discussed in Theorems A and B are completely decomposed as connected sums of many copies of $\mathbb{C}P^2$ and $\overline{\mathbb{C}P}^2$, and in particular, can also be written as sums of indefinite manifolds with $S^2 \times S^2$. Hence all results proved about spheres in a single class of given square, divisibility and type, hold for spheres in any class of the same square, divisibility and type.

3. Building spheres of higher self-intersection and divisibility

This section develops the tools needed to establish Theorems A and B. The key is to "embellish" a family of *unit spheres* (meaning embedded 2-spheres of square $+1$) in a 4-manifold X that satisfy the conclusions of Theorem A (such as those produced by the authors in [3]) to produce families of spheres in a suitable blowup of X of arbitrary nonnegative square that also satisfy these conclusions, or of square 0 and arbitrary divisibility that satisfy the conclusions of Theorem B. We first extend the notion of *surgery* along a sphere of square 0 to an operation "blowup-surgery" that applies to spheres of arbitrary square; this operation will be used to distinguish the embellished spheres.

Blowup-surgery.

Definition 3.1. *Blowup-surgery* on a 4-manifold X along an embedded 2-sphere S of square $n \geqslant 0$ is the operation of blowing up n points on S, and then surgering the resulting sphere of square 0. This produces a 4-manifold denoted X/S (see the next remark). There is an analogous operation for spheres of negative square, replacing the initial blowups with anti-blowups.

Remark 3.2. If S has square 0, then blowup-surgery is just regular surgery. If S has square ± 1, then blowup-surgery along S has the same effect as blowing down S, as shown in Figure 2, whence the common notation X/S for the result.

The families of homologous spheres constructed in the proofs of the theorems in the next section are distinguished by the Seiberg-Witten invariants of their blowup-surgeries. It follows, as shown in the next proposition, that any pair of such spheres of *nonzero* square n must intersect in *strictly more* than $|n|$ points; this complements the upper bound on their minimal geometric intersection number provided by Theorem A.

Proposition 3.3. *The minimal geometric intersection number of any pair of homologous* 2*-spheres S and T of nonzero square n in a* 4*-manifold X whose blowup-surgeries have distinct nontrivial Seiberg-Witten invariants must be strictly greater than $|n|$.*

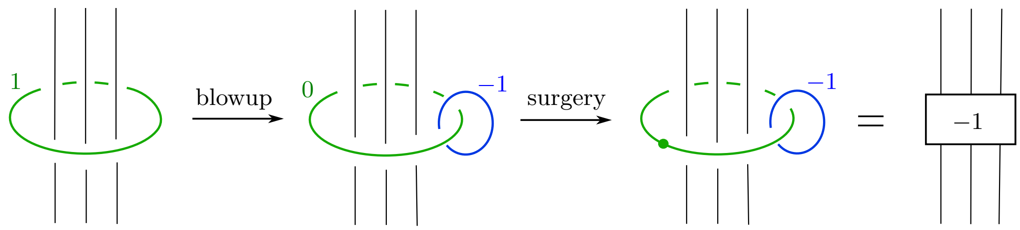

Figure 2. Blowdown as a blowup-surgery

Proof. We may assume $n > 0$, changing orientation if necessary. Now move S transverse to T so that $|S \cap T| = m \geqslant n$. If $m = n$, then blowing up $n-1$ of the points in $S \cap T$ produces two spheres P and Q of square $+1$ in $Y = X \# (n-1)\overline{\mathbb{C}P}^2$ that intersect in a single point. Let Q' denote the image of Q in the blowdown Y/P. By Remark 3.2, $Y/P \cong X/S$ and $Y/Q \cong X/T$, so by hypothesis Y/P and Y/Q have distinct Seiberg-Witten invariants. But Y/Q is obtained from Y/P by Gluck twisting Q', contradicting Lemma 2.3. Thus $m > n$. $\square$

Embellishing unit 2-spheres.

Fix a simpy-connected 4-manifold X, and set $X^i = X \# i\,\overline{\mathbb{C}P}^2$. Any *unit sphere* S in X (that is, an embedded 2-sphere of square $+1$) gives rise to two infinite sequences of simple 2-spheres

$$S^1, S^2, S^3, \ \dots \ \text{in } X^1 \qquad \text{and} \qquad S_0, S_1, S_2, \ \dots \ \text{in } X^2$$

where S^d has square 0 and divisibility d, while S_n has square n and represents a primitive ordinary class. These spheres, the *embellishments* of S, will be constructed inside a once or twice blown up tubular neighborhood of S.

Some care must be taken in defining these embellisments to insure that they depend up to isotopy only on the isotopy class of S (see Proposition 3.5a) and to arrange for corresponding embellishments of *families* of 2-spheres to lie in the same 4-manifold, not just diffeomorphic 4-manifolds. For these purposes, fix a pair of base points p and q in X to be blown up to give X^1 (blowing up p) and X^2 (blowing up p and q), and then proceed as follows:

1) Isotop S so that it contains q but not p.

2) Choose a tubular neighborhood N of S containing both basepoints, and identify N with $(\mathbb{C}P^2)^\circ$ (the $^\circ$ indicates that a small open 4-ball has been removed) in such a way that S corresponds to a (projective) line. This identifies $N^i = N \# i\,\overline{\mathbb{C}P}^2$ with $(\mathbb{C}P^2 \# i\,\overline{\mathbb{C}P}^2)^\circ$.

3) Choose a line T in N disjoint from the base points, and for each integer $d \geqslant 1$, a pencil P^d of d lines in N through p disjoint from q and $S \cap T$. The 2-dimensional (real) analogue of this configuration $\Gamma = S \cup T \cup P^d$ of 2-spheres is shown in Figure 3.

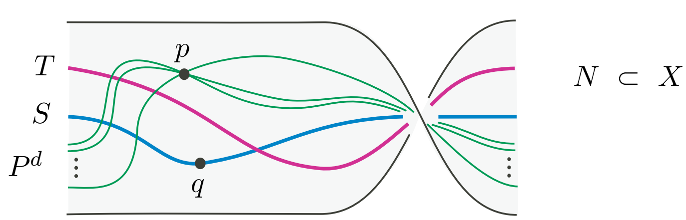

Figure 3. Configuration $\Gamma = S \cup T \cup P^d$ of projective lines in $N = (\mathbb{C}P^2)^\circ$ inside X

Now build S^d from the pencil P^d by blowing up p. When $d = 1$, this produces the 2-sphere $S^1 \subset X^1$ with trivial normal bundle. When $d > 1$, this yields d parallel copies of S^1 which can then be joined together (oriented consistently) using trivial tubes to give S^d. Next build S_{2d} from $S \cup P^d$ by blowing up both p and q and then *smoothing* the resulting configuration, i.e. replace pairs of transverse disks near the double points by annuli. The sphere S_{2d+1} is built in the same way from $T \cup P^d$. The results are shown in Figure 4 for the case $d = 2$.

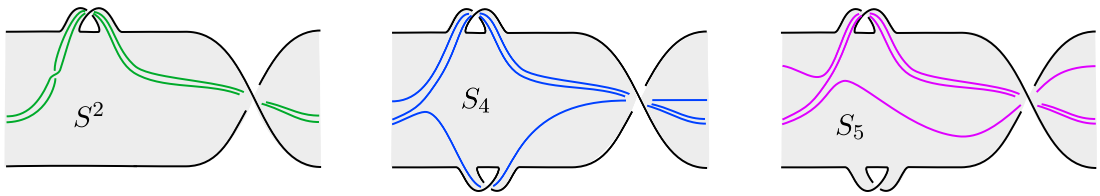

FIGURE 4. Some embellishments of S in $N = (\mathbb{C}P^2)^\circ$ inside X

Remarks 3.4. **a)** The tubular neighborhood N decomposes as a handlebody with one 0-handle (the part of N shown in Figure 3, containing all the multiple points of the configuration Γ) and one 2-handle, in which Γ appears (unaltered after the blowups) as parallel copies of the core of the handle. This provides a viewpoint that is useful when analyzing the interactions between embellishments of distinct unit spheres.

b) Another useful model for the embellishments of S arises from the well known fact that $\mathbb{C}P^2 \# 2\,\overline{\mathbb{C}P}^2$ is diffeomorphic to $\mathbb{H}_n \# \overline{\mathbb{C}P}^2$ for *any* integer n, where $\mathbb{H}_n$ is the 2-sphere bundle over the 2-sphere of Euler class n. Thus N^2 can be identified with $\mathbb{H}_n^\circ \# \overline{\mathbb{C}P}^2$, in fact in a way that identifies S_n with a *section* of $\mathbb{H}_n$; this can be seen for example using the Kirby calculus. For $d > 0$, we use a diffeomorphism $N^1 = (\mathbb{C}P^2 \# \overline{\mathbb{C}P}^2)^\circ \cong \mathbb{H}_1^\circ$ that identifies S^1 with a *fiber* of $\mathbb{H}_1$, so identifies S^d with d fibers of $\mathbb{H}_1$ tubed together.

With these models in hand, it is easy to establish the following properties of the embellishments of unit spheres:

Proposition 3.5. **a)** *Isotopic unit spheres in X have isotopic embellishments in $X^i = X \# i\,\overline{\mathbb{C}P}^2$.*

b) *The embellishments S_n and S^d of a unit sphere S in X are all simple (meaning that their complements have cyclic fundamental groups), and the S_n are also primitive and ordinary.*

c) *If S and T are unit spheres in X, then*

$$m_{X^1}(S^d, T^d) \leqslant m_X(S,T)\, d^2 \qquad \text{and} \qquad m_{X^2}(S_n, T_n) \leqslant m_X(S,T)\, (\lfloor n/2 \rfloor + 1)^2$$

where we write $m_Y(E,F)$ for the minimal geometric intersection number of any pair of surfaces E and F in a 4-manifold Y.

Proof. **a)** After positioning a unit sphere $S \subset X$ as in step 1) above, the tubular neighborhood theorem shows that the isotopy classes of its embellishments are not affected by the choices in the subsequent steps of the construction. The fact that these isotopy classes are not affected by the initial placement of S now follows from Lemma 2.1.

b) The sphere $S_n \subset X^2$ is simple, primitive and ordinary since it is geometrically dual to a sphere S^1 of square 0. To show that S^d is simple requires a little more work. Viewing $N^1 = \mathbb{H}_1^\circ$ as above, S^d is constructed by tubing together d copies $F_1 \ldots F_d$ of the fiber of $\mathbb{H}_1$. The complement of the union of the F_i in $\mathbb{H}_1$ is a trivial S^2-bundle over the d-punctured sphere, with free fundamental group of rank $d-1$. A natural presentation for this group is

$$(x_1, \ldots, x_d \mid x_1 \cdots x_d = 1)$$

where x_i is the meridian of F_i. Tubing F_i to F_{i+1} adds the relation $x_i = x_{i+1}$. Thus $\pi_1(\mathbb{H}_1 - S^d) = (x_1 \mid x_1^d = 1)$ is cyclic, and so $\pi_1(X^1 - S^d)$ is as well, by van Kampen's Theorem.

c) We may assume that S and T have been moved (as in the first step of the construction of their embellishments described above) to coincide along a 2-disk, and otherwise to intersect transversely in $m = m_X(S,T)$ points. We may also choose tubular neighborhoods that coincide along their 0-handles (as in Remark 3.4a) and whose 2-handles have m plumbed intersections. It is then clear that embellishments of S^d and T^d can be chosen that intersect in md^2 points, since each transverse double point in $S \cap T$ yields a grid of d^2 double points in $S^d \cap T^d$, which establishes the first inequality. The second follows by the same argument, with $\lfloor n/2 \rfloor + 1$ in place of d. □

Remark 3.6. The arguments given above for Proposition 3.5 work equally in the smooth and topological settings. Since the families of spheres of higher square and divisibility to be constructed in our proofs of Theorems A and B below will be embellishments of families of unit primitive spheres, pairwise topological isotopies for the latter easily yield the same for the former.

We now identify the manifolds produced by blowup-surgery along the embellishments of $S \subset X$ in terms of the blowdown X/S. Recall from Definition 3.1 and Remark 3.2 that blowup-surgery of a 4-manifold Y along a 2-sphere $T \subset Y$, also denoted Y/T, coincides with ordinary surgery when T has square 0, and with the blowdown of T when T has square ± 1.

Proposition 3.7. *Let S be a unit sphere in a simply-connected 4-manifold X, with embellishments $S^d \subset X^1$ and $S_n \subset X^2$, where $X^i = X \# i\,\overline{\mathbb{C}P}^2$ as above. Then the blowup-surgeries*

a) $X^1/S^d \;\cong\; X/S \# Q_d$

b) $X^2/S_n \;\cong\; X/S \# (n+1)\,\overline{\mathbb{C}P}^2$

where Q_d is a rational homology sphere with cyclic fundamental group. It follows that if T is another unit sphere in X such that the blowdowns X/S and X/T are distinguished by their Seiberg-Witten invariants, then the embellishments S_n and T_n are stably distinct for all n,† *as are S^d and T^d.*

Proof. **a)** Viewing $S^d \subset \mathbb{H}_1^\circ = N^1 \subset X^1$, as in the proof of Proposition 3.5b, we have

$$X^1/S^d \;=\; (X - N) \cup N^1/S^d \;\cong\; X/S \# Q_d$$

where $Q_d = \mathbb{H}_1/S^d$, the result of surgering $\mathbb{H}_1$ along S^d. But it was shown in that proof that the fundamental group of the complement $C = \mathbb{H}_1 - S^d$ is cyclic, and this group is isomorphic to $\pi_1(Q_d)$ by van Kampen's Theorem. It follows by a standard argument that Q_d is a rational homology 4-sphere. Indeed (using rational coefficients) $H_1(C) = 0$, so $H_2(C) = \mathbb{Q}$ by the sequence of the pair $(\mathbb{H}_1, C)$, whence $H_2(Q_d) = 0$ by the sequence of the pair (Q_d, C); it follows by duality that Q_d has the rational homology of a 4-sphere, as asserted.

b) Viewing S_n as a section of $\mathbb{H}_n$ in $N^2 = \mathbb{H}_n^\circ \# \overline{\mathbb{C}P}^2$, as discussed above Proposition 3.5, the calculation in Figure 5 shows $\mathbb{H}_n/S_n \cong n\,\overline{\mathbb{C}P}^2$. Thus $N^2/S_n = \mathbb{H}_n^\circ/S_n \# \overline{\mathbb{C}P}^2 \cong ((n+1)\overline{\mathbb{C}P}^2)^\circ$, so

$$X^2/S_n \;= (X - N) \cup N^2/S_n \;\cong\; X/S \# (n+1)\,\overline{\mathbb{C}P}^2$$

as asserted.

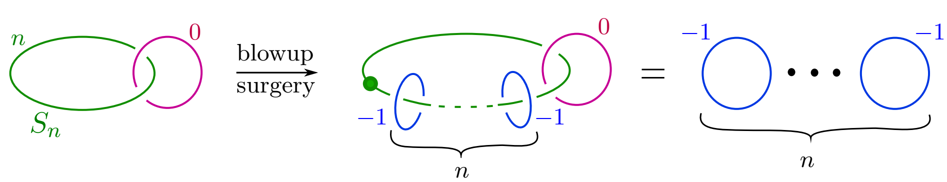

Figure 5. $\mathbb{H}_n/S_n \;\cong\; n\,\overline{\mathbb{C}P}^2$

† Recall that this means they are not related by a diffeomorphism, even after arbitrarily many external blowups

The last statement follows from the Seiberg-Witten blowup formula, and from an observation of Kotschick, Morgan and Taubes (implicit in [26, Proposition 2]) that 4-manifolds distinguished by their Seiberg-Witten invariants remain so after summing with any rational homology 4-sphere. $\square$

Internal stabilization.

Embedded 2-spheres S of arbitrary square in a simply-connected 4-manifold X give rise to sequences of surfaces in X

$$S(0) = S,\ S(1),\ S(2),\ \ldots$$

where $S(g)$ is of genus g, obtained from S by g successive internal stabilizations. The local nature of this stabilization operation shows that isotopies of S induce isotopies of $S(g)$ for every g. It is known that for homologous 2-spheres $S, T \subset X$ that are both simple (i.e. their complements have cyclic fundamental groups), $S(g)$ and $T(g)$ are isotopic for *some* g [7, Theorem 1], but general bounds on g have been hard to come by; it is even conceivable that $g = 1$ is always enough. To prove Theorems A and B, we will need to relate these bounds for a pair of unit spheres to the bounds for their embellishments. To state our result, let $g_X(S, T)$ denote the *minimal stabilization genus* os S and T, defined to be the smallest $g \geqslant 0$ such that $S(g)$ and $T(g)$ are isotopic in X.

Proposition 3.8. *If S and T are unit spheres in a simply-connected 4-manifold X, then*

$$g_{X^1}(S^d, T^d) \leqslant g_X(S,T)\, d \qquad \text{and} \qquad g_{X^2}(S_n, T_n) \leqslant g_X(S,T)\,(\lfloor n/2 \rfloor + 1)$$

where $X^i = X \# i\,\overline{\mathbb{C}P}^2$ as usual.

Proof. Let $g = g_X(S,T)$. Thus $S(g)$ and $T(g)$ are isotopic in X, and for the first inequality, it suffices to show that $S^d(gd)$ and $T^d(gd)$ are isotopic in X^1. The argument is similar to the proof of Proposition 3.5c. Position S and T to agree along a 2-disk D, and choose their tubular neighborhoods to have the same 0-handle B (as in Remark 3.4a) containing D as a proper unknotted disk. Thus S^d and T^d will agree inside the blowup B^1 of B at p, and will otherwise consist of d parallel copies of the cores D_S and D_T of their respective 2-handles.

Focusing on S for the moment, construct $S(g)$ by performing the g internal stabilizations along D_S, so that a tubular neighborhood of $S(g)$ can be seen as B with a generalized 2-handle H (diffeomorphic to the thickening of a genus g surface minus a disk) attached. Similarly there is a tubular neighborhood of $S^d(gd)$ consisting of a neighborhood of $S^d \cap B^1$ together with d parallel copies of H; this layering of the tubes (as shown in Figure 6) can be arranged since the stabilizations all correspond to the attachment of *trivial* 1-handles (see [9, 7]).

Now using an analogous model for $T(g)$ and $T^d(gd)$, any isotopy from $S(g)$ to $T(g)$ in X, chosen to leave B fixed by Lemma 2.1, can be mirrored to produce one from $S^d(gd)$ and $T^d(gd)$ in X^1. This establishes the first inequality in the proposition. As in the proof of Proposition 3.5c, the second inequality follows by essentially the same argument, with $\lfloor n/2 \rfloor + 1$ in place of d. $\square$

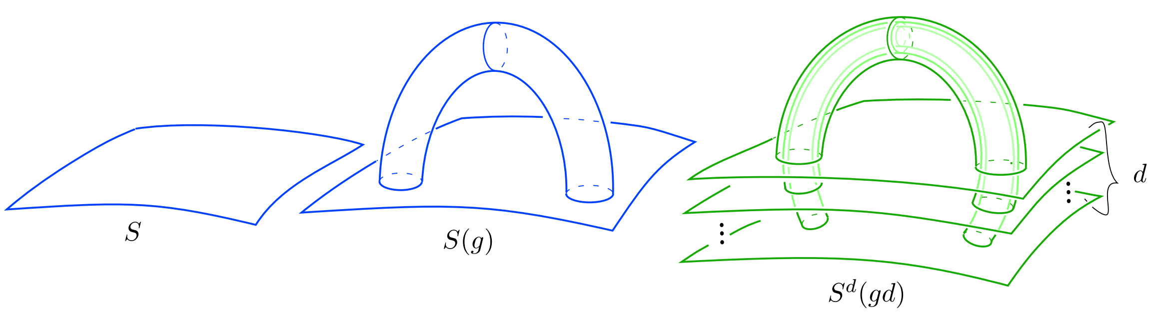

FIGURE 6. Layered tubes

Combining Propositions 3.5, 3.7 and 3.8, we immediately deduce:

Corollary 3.9. *Let S_i be any family of homologous unit spheres in a simply-connected 4-manifold X, distinguished by the Seiberg-Witten invariants of their blowdowns X/S_i and satisfying the three conditions* **1)**, **2)** *and* **3)** *in* Theorem A. *Set $X^i = X \# i\,\overline{\mathbb{C}P}^2$. Then for any nonnegative integer n, the spheres $S_{in} \subset X^2$ of square n satisfy the conclusion of* Theorem A, *and for any positive integer d, the spheres $S_i^d \subset X^1$ of square 0 and divisibility d satisfy the conclusion of* Theorem B.

4. Proofs of the theorems

For notational economy, set $\mathbb{X}_{p,q} = p\,\mathbb{C}P^2 \# q\,\overline{\mathbb{C}P}^2$, and assume $p, q \geqslant 2$. By Wall's Theorem 2.4, Theorem A holds for $\mathbb{X}_{p,q}$ provided it holds for a *single* primitive ordinary homology class in $\mathbb{X}_{p,q}$ of *each* nonnegative square, and in turn by Corollary 3.9, provided it holds for a single such class of square $+1$ in $\mathbb{X}_{p,q-2}$. Similarly Theorem B holds for $\mathbb{X}_{p,q}$ provided it holds for a single primitive ordinary class of square $+1$ in $\mathbb{X}_{p,q-1}$. Furthermore, if these theorems hold for $\mathbb{X}_{p,q}$ then they hold for $\mathbb{X}_{p,r}$ for any $r \geqslant q$, by the "stably distinct" condition in the statements of the theorems.

Thus Theorems A and B can be proved as stated by producing a *single* infinite sequence $S_1, S_2, \ldots$ of homologous unit spheres (that is, embedded 2-spheres of square $+1$) in each manifold of type

$$\text{(a)}\quad \mathbb{A}_k = \mathbb{X}_{2k,10k} \quad \text{for } k \geqslant 2 \qquad \text{and} \qquad \text{(b)}\quad \mathbb{B}_k = \mathbb{X}_{4k+3,20k+19} \quad \text{for } k \geqslant 1$$

that blow down to give 4-manifolds that are smoothly distinct (even after blowups) and that satisfy the three conditions in Theorem A. The authors' explicit construction in [3], coupled with the isotopy described in [7, §3.7], produced many such sequences S_i in any manifold of type $\mathbb{A}_k$, one for each sequence of knots K_i in the 3-sphere with distinct Alexander polynomials. In particular, the blowdowns $\mathbb{A}_k/S_i$ were shown in [3] to be diffeomorphic to $E(k)_{K_i} \# \overline{\mathbb{C}P}^2$, where $E(k)_K$ denotes Fintushel-Stern knot surgery [14] along a torus fiber in the elliptic surface $E(k)$. Since the Alexander polynomials of the K_i are distinct, the manifolds $\mathbb{A}_k/S_i$ have distinct Seiberg-Witten invariants [14, 47] and so are pairwise nondiffeomorphic, even after any number of blowups.

Note that the third condition in Theorem A for this family of spheres S_i is that their pairwise minimal geometric intersection numbers $m(S_i, S_j)$ should be at most 5. This fact is implicit in [3, Figure 13], and reduces to the claim that the two ribbon disks A_0 and B_0 shown there and reproduced in Figure 7 below[†] can be positioned rel boundary to intersect in at most 4 points. To see this, superimpose the immersed disks shown at the top of the figure. The result, drawn at the bottom, shows that these disks meet in four arcs, just as thickening a pair of clasped disks $D_1 \cup D_2$ bounded by the Hopf link yields four disks $(D_1 \times \partial I) \cup (D_2 \times \partial I)$ that intersect in four arcs. Pushing these disks into the 4-ball, one sees exactly one intersection point under each arc.

To produce a similar sequence of spheres in manifolds of type $\mathbb{B}_k$, the knots K_i must be chosen with more care. In particular, we require the blowdowns $\mathbb{A}_2/S_i$ of their associated knotted 2-spheres $S_i \subset \mathbb{A}_2$ (constructed in [3]) to have distinct "odd monopole counts", defined as follows:

Definition 4.1. The *odd monopole count* $\mathrm{SW}^{\mathrm{odd}}(X)$ of a closed 4-manifold X with $b_2^+(X) > 1$ is the number of Spin^c-structures on X with odd Seiberg-Witten invariant.

For example the $(2, 2i{-}1)$ torus knots work well since their Alexander polynomials $(t^{2i-1}+1)/(t+1)$ have exactly $2i{-}1$ non-zero coefficients, all odd, and there are two basic classes in $\mathbb{A}_2/S_i$ for each such coefficient, with Seiberg-Witten invariant equal to that coefficient [14]. Thus $\mathrm{SW}^{\mathrm{odd}}(\mathbb{A}_2/S_i) = 4i{-}2$.

[†] These disks are bounded by the same $(-3, 3, -3)$ pretzel knot K_0, and are obtained by pushing the immersed disks shown in the figure (related by a 180 degree rotation about the indicated horizontal axis) into the 4-ball. Adding a $+1$-framed 2-handle along K_0 produces the blowup of the classical Mazur cork, containing spheres S_0 and T_0, which can then be embedded in many different ways in the manifolds of type $\mathbb{A}_k$ and $\mathbb{B}_k$ to produce the spheres S_i.

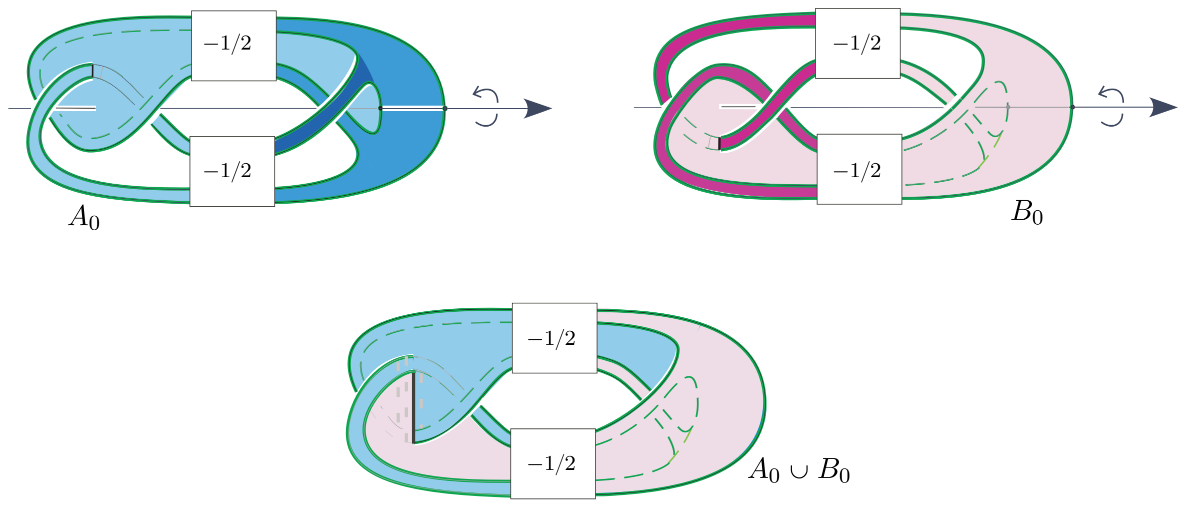

FIGURE 7. Ribbon disks

Now fix $k \geqslant 1$, and view the spheres S_i as lying in the $\mathbb{A}_2$−summand of the 4-manifold

$$\mathbb{B}_k \;=\; \mathbb{X}_{4k+3,20k+19} \;\cong\; \mathbb{A}_2 \# E(2k).$$

The last diffeomorphism exists since elliptic surfaces are almost completely decomposable [32]. To complete the proof it suffices to show that for distinct values of i, the blowdowns

$$\mathbb{B}_k/S_i \;\cong\; (\mathbb{A}_2/S_i) \# E(2k)$$

are smoothly distinct, since the other properties required of the spheres $S_i \subset \mathbb{B}_k$ are inherited from the corresponding properties of $S_i \subset \mathbb{A}_2$. But this follows from Bauer's calculation [6] of the Bauer-Furuta invariant [5] for connected sums, using the following special case of [6, Prop. 4.5]:

Theorem 4.2. (Bauer) *Let X and Y be simply-connected 4-manifolds of Seiberg-Witten simple type* (meaning that all their SW-basic classes have zero dimensional moduli spaces)*. Then the number of* Spin^c *structures on $X \# Y$ with nontrivial Bauer-Furuta invariant is equal to the product* $\mathrm{SW}^{\mathrm{odd}}(X)\,\mathrm{SW}^{\mathrm{odd}}(Y)$ *if $b_2^+(X) \equiv b_2^+(Y) \equiv 3 \pmod 4$, and is zero otherwise.*

Noting that $b_2^+(\mathbb{A}_2/S_i) = 3$ and $b_2^+(E(2k)) = 4k - 1 \equiv 3 \pmod 4$, and that $\mathrm{SW}^{\mathrm{odd}}(\mathbb{A}_2/S_i) = 4i - 2$ and $\mathrm{SW}^{\mathrm{odd}}(E(2k)) \neq 0$, it follows that the $\mathbb{B}_k/S_i$ for $i = 1, 2, \ldots$ have distinct numbers of nontrivial Bauer-Furuta invariants, so are smoothly distinct. □

Remarks 4.3. a) The family of manifolds with odd b_2^+ to which Theorems A and B apply can be extended as follows. In applying Bauer's Theorem 4.2, one could sum with a smooth hypersurface V_d of $\mathbb{C}P^3$ in place of an elliptic surface, with d chosen to be a multiple of 4 so that

$$b_2^+(V_d) \;=\; d(d-2)(d-4)/3 + d - 1 \;\equiv\; 3 \pmod 4$$

(see [18, Theorem 1.3.8] for the topological invariants of hypersurfaces). Now $V_d \cong E(2)$ when $d = 4$, and is of general type when $d > 4$, with Seiberg-Witten invariant 1 on the canonical Spin^c structure, so in all these cases $\mathrm{SW}^{\mathrm{odd}}(V_d)$ is nonzero. Using V_d in this way (for $d > 0$ divisible by 4) gives examples of families of spheres as in Theorem A in the manifolds $\mathbb{X}_{p,q}$ where

$$p \;=\; 4 + \tfrac{1}{3}(d^3 - 6d^2 + 11d - 3) \qquad \text{and} \qquad q \;\geqslant\; 20 + \tfrac{1}{3}(2d^3 - 6d^2 + 7d - 3).$$

Note that for large d, the manifolds $\mathbb{X}_{p,q}$ have $b_2^- \approx 2b_2^+$, rather than $b_2^- \approx 5b_2^+$, so although sparser than the families of 4-manifolds constructed in the proof above (p grows cubically rather than linearly), these $\mathbb{X}_{p,q}$ have smaller b_2^-/b_2^+.

b) Theorem B also holds for any nonprimitive homology class of type 0 and self-intersection $+4$ (or -4, by reversing orientations). Indeed, the associated primitive class can be represented by infinitely many spheres of self-intersection $+1$ as in Theorem A, each of which can be tubed to an oriented pushoff of itself (the local model for this is a smooth degree two curve in $\mathbb{C}P^2$) to produce the desired spheres representing the non-primitive class. To distinguish these spheres, one can either perform blowup-surgeries as before, or rational blow downs [13]. The stability properties are proved as in Corollary 3.9.

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