Topology Vol. 18, pp. 173–175

0040-9383/79/0301-0173/$02.00/0

BORDISM OF DIFFEOMORPHISMS

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(*Recieved* 14 *April* 1978)

IN THIS note we extend to the 3-dimensional case the results of M. Kreck on bordism of diffeomorphisms [3]. Some of this work is from the author's doctoral dissertation [5]. It is a pleasure to acknowledge my indebtedness to Robion Kirby for his insight and encouragement. We also thank the referee for a careful reading and many useful suggestions.

First recall the bordism relation for diffeomorphisms introduced by Browder in [1]. Two orientation preserving diffeomorphisms $h_i: M_i \to M_i$ of closed, oriented m-manifolds $M_i (i = 0, 1)$ are *bordant* if there is an oriented bordism W between M_0 and M_1 and an orientation preserving diffeomorphism $H: W \to W$ such that $H|M_i = h_i$. The collection of bordism classes forms an abelian group Δ_m under disjoint union.

In [2], after several partial results by Winkelnkemper [9] and Medrano [4], Kreck proved

THEOREM (Kreck). $\Delta_m = \Omega_m \oplus \hat{\Omega}_{m+1}$ (odd $m \neq 3$)

where Ω_* is the bordism group of oriented manifolds and $\hat{\Omega}_*$ is the kernel of the signature homomorphism. The even dimensional calculations ($m \neq 2$) were announced in [3].

We extend this theorem to the case $m = 3$. The case $m = 2$ remains open.

THEOREM. $\Delta_3 = 0$.

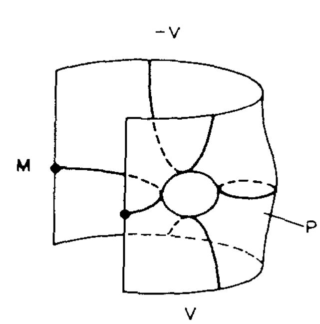
Before giving the proof, we introduce some notation. If V is a manifold and L is a framed link of spheres in V, then V/L will denote the manifold obtained by surgery on L in V. If V has boundary, then we identify ∂V with $\partial(V/L)$ in the obvious way.

Proof of theorem. Given an orientation preserving diffeomorphism $h: M \to M$ of a closed, oriented 3-manifold M, we must find a compact oriented 4-manifold W and an orientation preserving diffeomorphism $H: W \to W$ such that $\partial W = M$ and $H|M = h$.

It is well known that M bounds a compact, oriented, simply connected 4-manifold V. By connected summing with CP^2, if necessary, we may insure that the intersection form on $H_2(V)$ is odd.

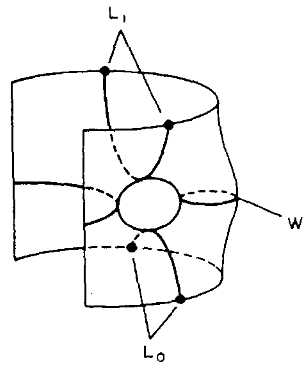
Form the closed 4-manifold $V \cup_h -V$ by identifying the boundaries of two copies of V by the diffeomorphism h. By Novikov additivity the signature of $V \cup_h -V$ is zero, and so it bounds a 5-manifold P. Suger P to make it simply connected.

Since M is bicollared in ∂P, we may view P as a relative bordism between V and itself, as indicated below.



Since P is simply connected and V is connected, $\pi_i(P, V) = 0$ $(i = 0, 1)$. The proof of the relative h-cobordism theorem goes through up to the middle dimensions to cancel all 0, 1, 4, and 5-handles on P (compare p. 146 in [8]).

The attaching maps of the 2-handles of P define a framed link L_0 of circles in V. Surgery on L_0 in V using the given framing yields the 4-manifold W which separates the 2 and 3-handles of P. Similarly, the attaching maps of the dual 2-handles (inverted 3-handles) of P define a framed link L_1 in V. Surgery on L_1 also yields W. In particular, there are diffeomorphisms h_i: $V/L_i \to W$ $(i = 0, 1)$ with $h_1^{-1}h_0|\partial V = h$.



As V is simply connected, surgery on a circle has the effect of connected summing with a 2-sphere bundle over the 2-sphere (see p. 135 in [7]), thus increasing the rank of $H_2(V)$ by two. It follows that L_0 and L_1 have the same number of components.

Since the intersection form on $H_2(V)$ is odd, we can change the framing on any circle by an isotopy (see discussion on pp. 134–135 and Lemma 4 of [7]). Thus there is a framed link L isotopic to both L_0 and L_1.

For $i = 0$ and 1, let f_i: $V \to V$ be the end of an isotopy which fixes ∂V and maps L to L_i as framed links. This induces a diffeomorphism g_i: $V/L \to V/L_i$ which is the identity on ∂V. Identifying V/L with W, we have

$$H: W \to W$$

given by $H = g_1^{-1}h_1^{-1}h_0g_0$, which restricts to h on $M = \partial W$. This proves the Theorem.

Remark 1. One may prove Kreck's result in the same way. The map

$$\Delta_m \to \Omega_m \oplus \hat{\Omega}_{m+1}$$
$$(M, h) \to ([M], [M_h])$$

where M_h is the mapping torus of h, is an epimorphism by a result of Neumann [6]. To show it is a monomorphism for odd $m = 2k - 1$, choose an $(m + 1)$-manifold V with odd intersection form on $H_k(V)$ and with $\partial V = M$ (M bounds), and an $(m + 2)$-manifold P with $\partial P = V \underset{h}{\cup} - V$ ($V \underset{h}{\cup} - V$ is cobordant to M_h, which bounds). Surgery below the middle dimension as in [2, §5] gives

(i) P and V are simply connected
(ii) $\pi_i(P, V) = 0$ for $i < k$
(iii) $\pi_k(P, V) \to \pi_{k-1}(V)$ is the zero map.

The proof of the relative h-cobordism theorem gives a handle-body structure for P relative to V with only k and $(k + 1)$-handles, and with the attaching $(k - 1)$-spheres trivial. As the intersection form on $H_k(V)$ is odd, we may proceed exactly as in the proof above.

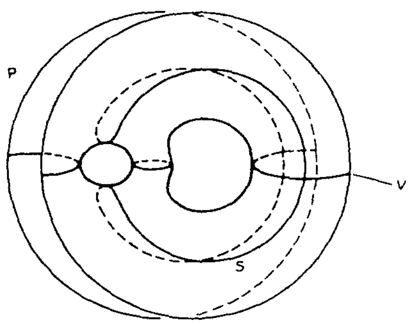
Remark 2. Our proof shows somewhat more than the stated theorem, namely that an orientation preserving diffeomorphism of the boundary of a simply connected 4-manifold V with odd intersection form extends to $V \# r(S^2 \times S^2)$ for some r. We simply arrange that the framing on each component of L (in the proof) is untwisted, giving $\# S^2 \times S^2$ after surgery.

Furthermore, we may drop the condition on the intersection form, showing that diffeomorphisms of 3-manifolds extend to parallelizable 4-manifolds. For (in the notation of the proof) if V is even, then P may be chosen even (Lemma 1 in [8]). Consequently L_0 and L_1 automatically have untwisted framings, and so are isotopic as framed links (cf. p. 147 in [8]). As oddness of the form was needed only for this fact, the result follows.

Remark 3. It follows from the proof of the theorem that any smooth map from a compact, oriented 5-manifold Q to the circle which is a fiber bundle projection on ∂Q is bordant (relative to ∂Q) to a fiber bundle projection.

First pull back a regular value of the given map $f: Q \to S^1$ to a 4-manifold V in Q. After a bordism of f (across appropriate 2-handles attached to $Q \times I$ along circles in $V \times 1$ and $(Q - V) \times 1$) we may assume that V and $Q - V$ are simply connected and that V has an odd intersection form. Cutting Q open along V gives a simply connected 5-manifold P which may be built on V using only 2 and 3-handles. As in the proof of the theorem, we may assume that the attaching links L_0 and L_1 (for the 2-handles and the dual 2-handles of P) coincide as framed links.

Consider the collection S of 2-spheres in Q made up of the cores of the 2-handles and the dual 2-handles in P. Since the framings on L_0 and L_1 match up, these 2-spheres have trivial normal bundles.



The reader may verify that surgery on S in Q yields a bundle over the circle. Extending f across the trace of this surgery gives the desired bordism.

As in Remark 1, a similar proof may be given for Q of arbitrary odd dimension (cf. Theorem 3 in [2]).

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