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Tori in the diffeomorphism groups of simply-connected 4-manifolds

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Let M be a closed simply-connected 4-manifold. All manifolds will be assumed smooth and oriented. The purpose of this paper is to classify up to conjugacy the topological subgroups of Diff (M) isomorphic to the 2-dimensional torus T^2 (Theorem 1), and to give an explicit formula for the number of such conjugacy classes (Theorem 2). Such a conjugacy class corresponds uniquely to a weak equivalence class of effective T^2 -actions on M. Thus the classification problem is trivial unless M supports an effective T^2 -action. Orlik and Raymond showed that this happens if and only if M is a connected sum of copies of $\pm \mathbb{C}P^2$ and $S^2 \times S^2$ (2), and so this paper is really a study of the different T^2 -actions on these manifolds.

1. Statement of results

An unoriented k-cycle $\langle e_1 \dots e_k \rangle$ is the equivalence class of an element (e_1, \dots, e_k) in \mathbb{Z}^k under the equivalence relation generated by cyclic permutations and the relation $(e_1, \dots, e_k) \sim (e_k, \dots, e_1)$.

From any unoriented cycle $\langle e_1 \dots e_k \rangle$ one may construct an oriented 4-manifold P by plumbing B^2 -bundles over S^2 according to the weighted circle shown in Fig. 1. Such a 4-manifold will be called a *circular plumbing*. The *core* of the plumbing is the union Sof the zero-sections of the constituent bundles. The diffeomorphism type of the pair (P, S) determines $\langle e_1 \dots e_k \rangle$, and so there is a one-to-one correspondence between circular plumbings with specified cores and unoriented cycles. Set

$$e(P,S) = \langle e_1 \dots e_k \rangle.$$

Now let M be a closed simply-connected 4-manifold, and T be a topological subgroup of Diff (M) isomorphic to T^2 . (The existence of such a subgroup implies that Mis the connected sum of copies of $\pm \mathbb{C}P^2$ and $S^2 \times S^2$.) We will associate with T an unoriented cycle e(T) as follows.

Let S(T) be the set of all points x in M for which there is a diffeomorphism $t \neq 1$ in T with tx = x. Denote a regular neighbourhood of S(T) by P(T). An educated glance at the orbit space of any T^2 -action associated to T establishes the following result (see § 3).

PROPOSITION. P(T) is a circular plumbing with core S(T). Now set

$$e(T) = e(P(T), S(T)).$$

A formula for e(T) will be given in §3.

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Definition. A reducible cycle is an unoriented cycle that can be reduced to $\langle 00 \rangle$ using the moves

 $\begin{array}{l} B_0: \langle \dots c \ 0 \ d \dots \rangle \rightarrow \langle \dots c + d \ \dots \rangle \\ B_{\pm 1}: \langle \dots c \ \pm 1 \ d \dots \rangle \rightarrow \langle \dots c \mp 1 \ d \mp 1 \dots \rangle. \end{array}$

Remark. Using the point of view of §2 it can be shown that an unoriented cycle is reducible if and only if its associated circular plumbing has boundary T^3 .

THEOREM 1. Let M be a connected sum of copies of $\pm \mathbb{C}P^2$ and $S^2 \times S^2$. Then the assignment $T \to e(T)$ sets up a one-to-one correspondence between conjugacy classes of topological subgroups of $\operatorname{Diff}(M)$ isomorphic to T^2 and reducible cycles $\langle e_1 \dots e_k \rangle$ satisfying

(1)
$$k = \chi(M)$$
,

(2)
$$e_1 + \ldots + e_k = 3\sigma(M),$$

(3) e_1, \ldots, e_k are all even if and only if M is a spin manifold (i.e. has no $\pm \mathbb{C}P^2$ factors).

Here $\chi(M)$ denotes the Euler characteristic of M, and $\sigma(M)$ the signature of M.

THEOREM 2. The number of conjugacy classes of topological subgroups of Diff(M) isomorphic to T^2 (for M as in Theorem 1) is finite if and only if $M = \pm (\underset{k=2}{\#} \mathbb{C}P^2)$, for some $k \geq 2$. In this case this number is

$$t(k) = \frac{1}{2k}c(k-1) + \frac{3}{4}c\left(\frac{k}{2}\right) + \frac{1}{2}c\left(\frac{k-1}{2}\right) + \frac{1}{3}c\left(\frac{k}{3}\right),$$

where c(n) is defined recursively for natural numbers n by

$$c(n) = \sum_{i=1}^{n-1} c(i) c(n-i)$$

c(1) = 1

and

The first few values of t(k) are as follows: t(k) = 1 for k < 6, t(6) = 3, t(7) = 4, t(8) = 12, t(9) = 27, t(10) = 82, t(11) = 228, t(12) = 733.

c(x) = 0 otherwise.

Remark. It was brought to my attention by Craig Squier that the numbers c(n) are the Catalan numbers

$$c(n)=\frac{1}{2n-1}\binom{2n-1}{n}.$$



Fig. 2. Diagram of $PSL(2, \mathbb{Z})$.

This formula is obtained from the recursive definition in Theorem 2 as follows. The generating function

$$f(x) = \sum_{n=1}^{\infty} c(n) x^n$$

for c(n) evidently satisfies $f^2 = f - x$, and so

$$f(x) = \frac{1 - \sqrt{(1 - 4x)}}{2}$$

by the quadratic formula (with the minus sign chosen to satisfy c(1) = 1). The Taylor series for this function yields the formula.

2. The diagram of $PSL(2,\mathbb{Z})$

The well known diagram in Fig. 2 will be useful as a book-keeping device. It is obtained as follows.

First define a 1-complex K. The vertices of K are the points of $\mathbb{Q}P^1 = \mathbb{Q} \cup \{1/0\}$. Two vertices (p,q) and (p',q') are joined by an edge if and only if

$$\det egin{pmatrix} p & p' \ q & q' \end{pmatrix} = \pm \, \mathbf{1}.$$

Thus the oriented edges of K can be identified with the elements of $PSL(2, \mathbb{Z})$.

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Now embed K in the Poincaré disc D as the 1-skeleton of a tesselation by ideal triangles, obtained by successive reflections in the sides of one such triangle. The labelling of vertices is unique once it is given on one triangle. This labelled subcomplex $K \subset D$ is called the *Diagram of PSL*(2, \mathbb{Z}) (see Fig. 2).

Observe that the set $\mathbb{Q}P^1$, viewed as lines in \mathbb{R}^2 through 0 of rational slope, is invariant under the action of $GL(2,\mathbb{Z})$ on \mathbb{R}^2 . This gives an action of $GL(2,\mathbb{Z})$ on the vertices of K, which readily extends to an action on D by hyperbolic isometries leaving K invariant. The subgroup $SL(2,\mathbb{Z})$ acts by orientation preserving isometries.

Definition. A loop in K is a closed, oriented edge path in $K \subset D$, and will be identified by its (oriented) cycle $(p_1/q_1 \dots p_k/q_k)$ of vertices. If $p_i/q_i \neq p_j/q_j$ for $i \neq j$, then the loop is simple.

Two loops in K are conjugate if they lie in the same orbit of the action of $GL(2,\mathbb{Z})$ on the set of loops given by

$$A(p_1/q_1...p_k/q_k) = \begin{cases} (Ap_1/q_1...Ap_k/q_k) & \det A = 1\\ (Ap_k/q_k...Ap_1/q_1) & \det A = -1 \end{cases}$$

In order to study the conjugacy class of a loop $r = (p_1/q_1 \dots p_k/q_k)$ in K, associate with r an unoriented cycle

$$e(r) = \langle e_1 \dots e_k \rangle,$$

where

$$(*) e_i = \det\left(\begin{pmatrix} p_{i-1} & p_i \\ q_{i-1} & q_i \end{pmatrix} \begin{pmatrix} p_i & p_{i+1} \\ q_i & q_{i+1} \end{pmatrix} \begin{pmatrix} p_{i-1} & p_{i+1} \\ q_{i-1} & q_{i+1} \end{pmatrix} \right) \quad (i = 1, ..., k),$$

with indices taken mod k. In geometric terms, the loop turns across e_i triangles at the vertex p_i/q_i , with $e_i < 0$ if and only if the turn is clockwise.

LEMMA. (1) Two loops in K are conjugate if and only if their associated unoriented cycles are equal.

(2) An unoriented cycle is associated with a loop in K if and only if it is reducible.

Proof. (1) The direct implication is geometrically evident as $GL(2, \mathbb{Z})$ acts on (D, K) by isometries. Alternatively, it can be verified from (*).

The converse follows from the transitivity of the action of $SL(2,\mathbb{Z})$ on oriented edges in K.

(2) Any loop r in K can be collapsed to a vertex using elementary collapses C_0 and $C_{\pm 1}$ defined in Fig. 3. For r always contains a simple subloop r' (possibly = r). If r' is just a retraced edge, then C_0 can be applied. Otherwise r' encloses a triangulated polygon P in D. Viewing P as a regular Euclidean polygon, the shortest interior edge in P cuts off a triangle to which $C_{\pm 1}$ can be applied.

Now elementary collapses C_0 and $C_{\pm 1}$ on r correspond to moves B_0 and $B_{\pm 1}$ on e(r) (see § 1). Thus e(r) is reducible.

Conversely if e is a reducible cycle then the sequence of reducing operations corresponds to a sequence of elementary collapses, which in reverse gives instructions for building up a loop r in K with e(r) = e.

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Fig. 3. (A) C_0 : Elementary collapse of a retraced edge $(\dots p/q \ p'/q' \ p/q \dots) \rightarrow (\dots p/q \dots)$. (B) $C_{\pm 1}$: Elementary collapse across a triangle $(\dots p/q \ (p \pm p')/(q \pm q') \ p'/q' \dots) \rightarrow (\dots p/q \ p'/q' \dots)$ ($\dots p/q \ p'/q' \dots$) ($C_{\pm 1}$ shown).

3. Proofs

Let M be a connected sum of copies of $\pm \mathbb{C}P^2$ and $S^2 \times S^2$, and $\alpha: T^2 \to \text{Diff}(M)$ be an effective T^2 -action on M. Following Orlik and Raymond(2), the weighted orbit space of α is an oriented 2-disc with boundary consisting of singular orbits and interior consisting of principal orbits (Fig. 4). The k distinguished points on the boundary are fixed points, and the arcs joining them correspond to 2-spheres in M. (The Proposition in §1 follows readily.) The weights in $\mathbb{Q} \cup \{1/0\}$ specify the isotropy subgroups of these 2-spheres. In particular p/q specifies the circle in $T^2 = \mathbb{R}^2/\mathbb{Z}^2$ which is covered by the line in \mathbb{R}^2 through 0 of slope p/q.

As the weights satisfy

$$\det \begin{pmatrix} p_i & p_{i+1} \\ q_i & q_{i+1} \end{pmatrix} = \pm 1 \quad (i = 1, ..., k)$$

(see p. 534 in (2)), with subscripts taken mod k, the cycle

$$r(\alpha) = (p_1/q_1 \dots p_k/q_k)$$

represents a loop in K (see § 2). In fact, the equivariant classification theorem of Orlik and Raymond (2) shows that the assignment

$$\alpha \rightarrow r(\alpha)$$

sets up a one-to-one correspondence between equivalence classes of effective T^2 -actions on closed simply-connected 4-manifolds and loops in K. (Recall that two actions $\alpha_i: T^2 \to \text{Diff}(M_i) \ (i = 0, 1)$ are equivalent if there is a diffeomorphism $h: M_0 \to M_1$ with $h\alpha_0(g) = \alpha_1(g) h$ for all g in T^2 .)

Remarks. (1) Let S_i be the 2-sphere in M with weight p_i/q_i , and E_i be its normal



bundle in M. The T^2 -action on ∂E_i identifies ∂E_i as the lens space $L(e_i, 1)$ where e_i is given by the formula (*) in § 2. (See p. 550 in (2).) Thus E_i has Euler class e_i . Setting

$$T(\alpha) = \alpha(T^2)$$

$$e(T(\alpha)) = e(r(\alpha)).$$

(2) Recall that two actions $\alpha_i: T^2 \to \text{Diff}(M)$ (i = 0, 1) are weakly equivalent if α_1 and $\alpha_0 A$ are equivalent for some automorphism A of T^2 . Since Aut $T^2 = GL(2, \mathbb{Z})$, it is straightforward to show that α_0 and α_1 are weakly equivalent if and only if either of the following conditions is satisfied:

(a) $T(\alpha_0)$ and $T(\alpha_1)$ are conjugate in Diff (M),

(b) $r(\alpha_0)$ and $r(\alpha_1)$ are conjugate loops in K.

Proof of Theorem 1. By the Remarks above and the Lemma in §2, the assignment $T \rightarrow e(T)$ sets up a one-to-one correspondence between conjugacy classes of toral subgroups of the diffeomorphism groups of closed simply-connected 4-manifolds and reducible cycles. Since connected sums of copies of $\pm \mathbb{C}P^2$ and $S^2 \times S^2$ are classified by their Euler characteristics and the signatures and parities of their intersection forms, it remains to show

(1) $\chi(M) = k$,

(2) $\sigma(M) = (e_1 + \ldots + e_k)/3$, and

(3) M is spin if and only if e_1, \ldots, e_k are all even

for any effective T^2 -action α on M with associated reduced cycle $e(\alpha) = \langle e_1 \dots e_k \rangle$.

A well known theorem of P. Conner states that $\chi(M)$ is equal to the number of fixed points of α , giving (1).

To prove (2), observe that the moves B_0 and $B_{\pm 1}$ reducing $e(\alpha)$ (see § 1) have the following effect on the underlying manifold:

- B_0 surger a 2-sphere in M,
- $B_{\pm 1}$ blow down a 2-sphere in *M* of self intersection ± 1 .

Thus B_0 does not affect the signature of the underlying manifold, and $B_{\pm 1}$ alters it by ∓ 1 . An inductive argument, starting with $\sigma(S^4) = 0$ (corresponding to the reduced cycle $\langle 0 0 \rangle$), establishes (2).

(3) follows from Theorem 6.3 in (1).



Proof of Theorem 2. First observe that $S^2 \times S^2$ and $S^2 \times S^2 = \mathbb{C}P^2 \# - \mathbb{C}P^2$ each has an infinite number of conjugacy classes of toral subgroups in its diffeomorphism group, corresponding to the reducible cycles $\langle n \ 0 - n \ 0 \rangle$ with *n* even and odd, respectively. It follows that if *M* has a connected sum factor diffeomorphic to $S^2 \times S^2$ or $S^2 \times S^2$, then Diff (*M*) has infinitely many inconjugate toral subgroups.

Now assume that M has no $S^2 \times S^2$ or $S^2 \approx S^2$ factors. Then M or -M is a connected sum of copies of $\mathbb{C}P^2$. Without loss of generality we assume henceforth that

$$M = \underset{k-2}{\#} \mathbb{C}P^2$$

for some $k \ge 2$. Then for any toral subgroup T of Diff(M), e(T) is reducible to $\langle 0 0 \rangle$ using only moves B_1 . For, as observed in the proof of Theorem 1, move B_0 splits off a $S^2 \times S^2$ or $S^2 \approx S^2$ factor from M, and $B_{\pm 1}$ splits off a $\pm \mathbb{C}P^2$ factor.

It follows from the definition of move B_1 that Diff (M) has only a finite number t(k) of conjugacy classes of toral subgroups. These correspond exactly to actions associated with simple loops in K of length k. Any such loop r encloses an oriented, triangulated k-gon in D, whose regular Euclidean model we denote by P(r). Evidently the conjugacy class of r in K (or equivalently of the associated torus in Diff (M)) is uniquely determined by P(r) up to rigid Euclidean motions (including orientation reversing motions). In particular

t(k) = the number of triangulations of the k-gon (with no added vertices) up to rotations and reflections.

For example, the 6-gon has three triangulations (Fig. 5) and so Diff ($\# \mathbb{C}P^2$) has three conjugacy classes of toral subgroups.

To compute t(k), we first ignore reflectional symmetry and compute the number s(k) of triangulations of the k-gon up to rotations.

Observe that every triangulation of a regular polygon P as above has at most 3-fold rotational symmetry. (This was pointed out to me by Benjamin Halpern.) For, the centre of P lies either on an edge or inside a triangle. There can be at best 2-fold rotational symmetry in the former case, and 3-fold in the latter. See Fig. 6 (the circle represents P).



Let $s_i(k)$ denote the number of triangulations of the k-gon (up to rotations) which have *i*-fold rotational symmetry (for i = 1, 2, or 3). Note that

$$s(k) = \sum_{i=1}^{3} s_i(k)$$

Let C(n) denote the set of all simple edge paths in K of length n joining 1/0 to 0/1and lying in the upper half of the Poincaré disc D (i.e. with second vertex p/1 for some p > 0). Set

$$c(n) = \operatorname{card} C(n).$$

For n > 1, every path in C(n) must pass through 1/1, and so

$$c(n) = \sum_{i=1}^{n-1} c(i) c(n-i)$$

in agreement with the definition in the statement of the theorem.

For any loop r in K of length k, let S(r) be the set of all edge paths in C(k-1) with vertices $r_1 = 1/0, r_2, \ldots, r_k = 0/1$ such that r and $(r_1 \ldots r_k)$ lie in the same orbit of the action of $SL(2, \mathbb{Z})$ on K. Evidently the sets S(r) partition C(k-1) into disjoint subsets. Observe that if P(r) has *i*-fold symmetry (i = 1, 2, or 3), then S(r) contains k/i paths. Thus

$$c(k-1) = ks_1(k) + \frac{1}{2}ks_2(k) + \frac{1}{3}ks_3(k).$$

The triangulations with 2- and 3-fold rotational symmetry are evidently of the form



indicated in Fig. 7. One easily deduces that $s_i(k) = c(k/i)$ for i = 2 and 3. Since $s_1(k) = s(k) - s_2(k) - s_3(k)$, the equation displayed above yields

$$s(k) = \frac{1}{k}c(k-1) + \frac{1}{2}c\left(\frac{k}{2}\right) + \frac{2}{3}c\left(\frac{k}{3}\right).$$

Now we wish to compute t(k). The computation of s(k) ignored reflectional symmetry, and so we effectively counted triangulations without reflectional symmetry twice. Thus, setting r(k) equal to the number of triangulations with reflectional symmetry, we have

$$t(k) = \frac{1}{2}(s(k) + r(k)).$$

But r(k) is easy to compute by hand. For the triangulations with reflectional symmetry are of one of the three forms indicated in Fig. 8 (the dotted line is the line of reflection; \bar{x} represents the reflection of the edge path x through the edge joining 1/0 and 0/1). The number of triangulations (up to rotations and reflections) in each of the first two cases is

$$rac{1}{2}\left(c\left(rac{k}{2}
ight)+c\left(rac{k}{4}
ight)
ight).$$

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These two cases overlap in triangulations of the form shown in Fig. 9. There are c(k/4) such triangulations and so there are a total of c(k/2) triangulations in the first two cases of Fig. 8. In the third case there are c((k-1)/2) triangulations. Thus

$$r(k) = c\left(\frac{k}{2}\right) + c\left(\frac{k-1}{2}\right)$$

Combining this with the formula for s(k) above, we obtain the desired formula for t(k).

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