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4-DIMENSIONAL ORIENTED BORDISM

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In 1952 Rohlin [4] (see appendix) outlined a proof of the following result:

THEOREM. *Every closed oriented smooth 4-manifold M of signature zero is the boundary of a compact oriented smooth 5-manifold.*

Two years later Thom [6] gave a proof using stable homotopy theory as part of his general program for computing the oriented bordism groups. Although his methods are of fundamental importance, the proof is unnecessarily complicated in this particular case.

In a lecture at IHES in 1976, John Morgan proposed a more geometric proof of the theorem. (A sketch is given in Remark 1.) Morgan's proof followed Rohlin's outline, but used a fact not known to Rohlin: a simply connected cobordism of dimension ≥ 6 has a handlebody structure which reflects its homology structure [5].

We present a new proof of the theorem, also following Rohlin's outline. Our proof partially incorporates Morgan's (step 2 below) but avoids the handlebody theorem by using the Whitney immersion theorem and a transversality argument (steps 1 and 3). This is perhaps closer to what Rohlin had in mind.

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PROOF OF THE THEOREM. We shall work in the smooth category.

Observe that M is bordant to a simply connected manifold, obtained for example by surgery on a set of normal generators of the fundamental group of M [2]. So we may assume that M is simply connected.

STEP 1. *Find a (embedded) submanifold M_1 of S^7 which is bordant to M.*

By a theorem of Whitney [7] M immerses in S^7 with singular set consisting of double circles at which the sheets of M meet transversely. Each double circle C may be eliminated at the cost of a surgery on M, as follows. Since M is orientable, C is the image of two circles C_1 and C_2 in M (rather than one circle by a double cover). As M is simply connected, C_1 bounds a disc D (embedded in M) missing the rest of the singular set, so in fact D is

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embedded in S^7. D has a tubular neighborhood $D \times B^5$ in S^7 which intersects M in tubular neighborhoods $D \times (B^2 \times 0)$ of D and $\partial D \times (0 \times B^3)$ of C_2. Now remove $\partial D \times (0 \times B^3)$ from M and replace it with $D \times (0 \times \partial B^3)$. See Figure 1.

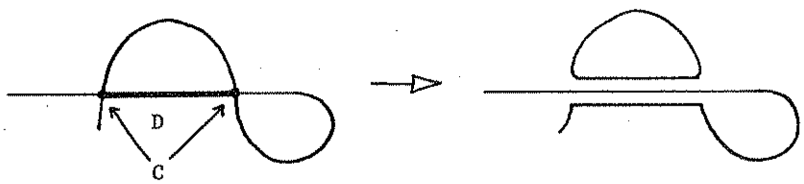

Figure 1

This leaves a simply connected 4-manifold bordant to M and immersed in S^7 with fewer double curves. The bordism is across the 2-handle $D \times (0 \times B^3)$. Continuing in this way we obtain M_1 bordant to M and embedded in S^7.

STEP 2. Let N be a tubular neighborhood of M_1 in S^7, and $W = S^7 - \mathrm{int}(N)$. Find a submanifold M_2 of ∂W which is null homologous in W and is diffeomorphic to $M_1 \# n\mathbb{C}P^2$ (for some integer n).*

We shall denote by $[Q]$ the class in $H_4(W)$ represented by a closed 4-manifold Q embedded in ∂W. The null-homologous condition above means that $[M_2] = 0$.

The geometric key to this step is the following observation of Morgan.

LEMMA. Let $p: E \to B$ be the trivial S^2-bundle over a 4-ball B. Then the image of any partial section $s: \partial B \to E$ bounds a submanifold K of E which is diffeomorphic to $k\mathbb{C}P^2 -$ (open 4-ball) (for some integer k).

PROOF. Using a trivialization, identify $p: E \to B$ with the projection map $p_1: B^4 \times S^2 \to B^4$.

Let $h: S^3 \to S^2$ be the Hopf map. Observe that the image of the partial section $S^3 = \partial B^4 \to B^4 \times S^2$ given by $x \to (x, h(x))$ bounds a Hopf disc bundle

$$H = \{(tx, h(x)) : t \in [0,1],\ x \in S^3\}$$

in $B^4 \times S^2$.

Now let k be the Hopf degree of the map $p_2 s: S^3 \to S^2$, where $p_2: B^4 \times S^2 \to S^2$ is the projection map. We may assume $k > 0$. (If $k < 0$ then the argument is analogous, and if $k = 0$ then s extends to a global section $t: B^4 \to B^4 \times S^2$ and so we may take $K = t(B^4)$.) Let B_i $(i = 1, \ldots, k)$ be disjoint 4-balls in B^4. Choose trivializations $T_i: B^4 \times S^2 \to p_1^{-1}(B_i)$ covering diffeomorphisms $t_i: B^4 \to B_i$. Then s extends to a partial section

*If $n > 0$ then nM denotes the connected sum of n copies of M. If $n < 0$ then $nM \equiv (-n)(-M)$, where $-M$ is M with the opposite orientation. Finally $0M \equiv S^4$.

$$t : B^4 - \left(\bigcup_i \operatorname{int}(B_i) \right) \longrightarrow B^4 \times S^2$$

with $p_2 T_i^{-1} t(t_i | S^3) = h$ for all i. Each 3-sphere $t(\partial B_i)$ bounds a Hopf bundle $H_i = T_i(H)$ in $B^4 \times S^2$. Set

$$K = \operatorname{im}(t) \cup \left(\bigcup_i H_i \right) .$$

<u>PROOF OF STEP 2</u>. A straightforward computation shows that the Euler class of the bundle

$$p : N \to M_1$$

is zero [3,§11.4]. It follows that there is a partial section

$$s : M_1 - \operatorname{int}(B) \to \partial N$$

where B is a 4-ball in M_1 [3,§12.5]. The lemma provides a submanifold $K = k\mathbb{C}P^2$ - (open 4-ball) of $p^{-1}(B) \cap \partial N$ with boundary $s(\partial B)$. Thus

$$L = \operatorname{im}(s) \cup K$$

is a submanifild of $\partial N = \partial W$ diffeomorphic to $M_1 \# k\mathbb{C}P^2$. See Figure 2.

The Euler class in H^3(M) is the P-dual of the class in H_1(M) = 0 represented by the intersection of M with a push off

Geometric way to see this: The normal bundle N has a section over the complement in M of a circle C representing the P-dual to the Euler class, and thus over any embedded disk D bounded by C in M (which is 1-connected) so just let B be a tub. nbd. of D in M

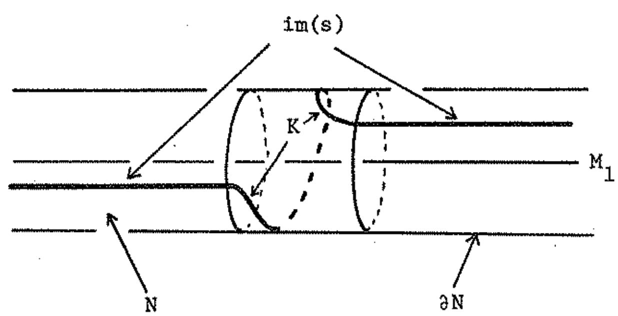

Figure 2

If $[L] = 0$ in $H_4(W)$ then we may take $M_2 = L$. So assume $[L] \neq 0$. Consider the isomorphism $d : H_4(W) \to H_2(M_1)$ defined by the commutative diagram

$$\begin{array}{ccc} H_4(W) & \xrightarrow{d} & H_2(M_1) \\ \partial \uparrow & & \uparrow \text{Thom isomorphism} \\ H_5(S^7, W) & \xleftarrow[\text{excision}]{} & H_5(N, \partial N) \end{array}$$

Represent $d([L])$ by an embedded surface F in $M_1 - B$. (One may think of F as the intersection of M_1 with a 5-~~cycle~~ chain bounded by L in S^7.) To get M_2, we shall modify L near $s(F)$.

Let D be a 4-ball in $M_1 - B$ which intersects F in a trivial 2-disc. Set $M_0 = M_1 - \mathrm{int}(D)$. Then $F_0 = F \cap M_0$ is a surface with boundary, properly embedded in M_0 (Figure 3).

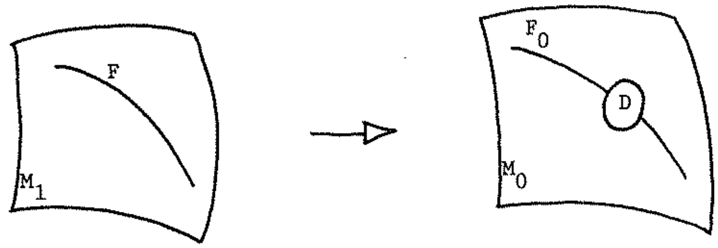

Figure 3

As F_0 is homotopy equivalent to a 1-complex, it has a tubular neighborhood $F_0 \times B^2$ in M_0, and N restricts to a trivial bundle over $F_0 \times B^2$. Pick a trivialization $(F_0 \times B^2) \times B^3$ (we suppress the map) so that $s(x,y) = (x,y,n)$ (n = the north pole of B^3) for all (x,y) in $F_0 \times B^2$.

Define a partial section

$$t: F_0 \times B^2 \to \partial N$$

by $t(x,y) = (x,y,f(y))$, where $f: B^2 \to S^2$ wraps B^2 around S^2 (i.e. $f(\partial B^2) = n$, $f(0) = -n$, and $f|\mathrm{int}(B^2)$ is an embedding). By the lemma, there is a submanifold $J = j\mathbb{C}P^2$ - (open 4-ball) of $p^{-1}(D) \cap \partial N$ with boundary $t(\partial D)$. Set

$$M_2 = (L - s(F_0 \times B^2 \cup D)) \cup t(F_0 \times B^2) \cup J .$$

M_2 is a submanifold of ∂W diffeomorphic to $M_1 \# (j+k)\mathbb{C}P^2$.

It remains to show that $[M_2] = 0$ in $H_4(W)$. Put $N_0 = p^{-1}(M_0)$ and $C = p^{-1}(D)$. Let X be the union of the straight line segments in each B^3 fiber joining $s(x,y)$ to $t(x,y)$, for (x,y) in $F_0 \times B^2$.

X can be extended across C to a 5-cycle $\bar{X}$ in N whose boundary represents $[L]-[M_2]$ in $H_4(W)$. Furthermore $\bar{X}$ intersects M_1 in F, and so $d([M_2]) = d([L]) - d([L]-[M_2]) = [F]-[F] = 0$ in $H_2(M_1)$. Thus $[M_2] = 0$ in $H_4(W)$.

As $\bar{X}$ is perhaps hard to visualize, we provide an alternative algebraic argument that $[M_2] = 0$. Consider the isomorphism $c: H_4(W \cup C) \to H_2(M_0, \partial M_0)$

defined by the commutative diagram

$$\begin{array}{ccc} H_4(W\cup C) & \xrightarrow{c} & H_2(M_0,\partial M_0) \\ \partial\uparrow & & \uparrow \text{ Thom isomorphism} \\ H_5(S^7,W\cup C) & \xrightarrow[\text{excision}]{} & H_5(N_0,\partial N_0)\ . \end{array}$$

One readily verifies that X represents the element of $H_5(N_0,\partial N_0)$ corresponding to $[L]-[M_2]$ in $H_4(W\cup E)$. Since X and M_0 intersect in F_0, $c([L]-[M_2]) = [F_0]$. It follows that $d([L]-[M_2]) = [F]$, by the commutativity of the following diagram

$$\begin{array}{ccccc} H_4(W) & & \xrightarrow{d} & & H_2(M_1) \\ \cong\downarrow & & & & \downarrow\cong \\ H_4(W\cup C) & \xrightarrow[c]{} & H_2(M_0,\partial M_0) & \xrightarrow[\text{excision}]{} & H_2(M_1,D)\ . \end{array}$$

Thus $[M_2]=0$ by the same argument as above.

STEP 3. Show that M_2 bounds a 5-manifold V.

This will prove the theorem. For then $\sigma M_2 = 0$. But $\sigma M_2 = \sigma M_1 + n$ (by step 2) and $\sigma M_1 = \sigma M = 0$ (by step 1), so $n=0$. Thus M_2 is bordant to M, and so M bounds.

To prove that M_2 bounds, first construct a map

$$f: W \to \mathbb{C}P^n$$

(for large n) with $f \pitchfork \mathbb{C}P^{n-1}$ and $f^{-1}(\mathbb{C}P^{n-1}) = M_2$. For example, define f on an open tubular neighborhood U of M_2 in ∂W to be the classifying map $U \to \mathbb{C}P^n - x$ of the normal bundle of M_2 in ∂W. (Here x is a point in $\mathbb{C}P^n$, and so $\mathbb{C}P^n - x$ is the canonical complex line bundle over $\mathbb{C}P^{n-1}$.) Extend f to ∂W by mapping $\partial W - U$ to x.

The only obstruction to extending f to a map

$$F: W \to \mathbb{C}P^n$$

lies in $H^3(W,\partial W;\pi_2(\mathbb{C}P^n))$. Since $\pi_2(\mathbb{C}P^n) = \mathbb{Z}$ is generated by a $\mathbb{C}P^1$ intersecting $\mathbb{C}P^{n-1}$ transversely in one point, this obstruction is Poincaré dual to $[M_2] \in H_4(W)$. Since $[M_2]=0$, F exists.

Now homotop $F(\text{rel } \partial W)$ transverse to $\mathbb{C}P^{n-1}$ and set

$$V = F^{-1}(\mathbb{C}P^{n-1})\ .$$

The proof is complete.

REMARK 1. Morgan's proof follows the same three step outline, but the proofs of steps 1 and 3 are different. Here is a sketch.

To achieve step 1, first embed M in S^8 by the Whitney embedding theorem. Using a normal vector field, push M out to the boundary of a tubular neighborhood N. Set $W = S^8 - \mathrm{int}(N)$. Since M may be taken simply connected (as in step 1 above), $H_*(W, \partial W)$ vanishes except in dimensions 3, 5 and 8. Build W as a handlebody on ∂W with handles of index 3,5 and 8. M misses the attaching 2-spheres of the 3-handles by general position, but may meet the attaching 4-spheres of the 5-handles in circles. Surgery on M (as in the proof of step 1 above) produces M_1 missing these as well. Thus M_1 lies in the boundary S^7 of an 8-handle.

Step 3 is similar to step 1. The only difficulty is in pushing M_2 off of the attaching 2-spheres of the 3-handles. But there is no algebraic obstruction to doing this since M_2 is null homologous in W, and so the Whitney trick applies. Finally we have M_3 (bordant to M_2 after pushing past the 5-handles) lying in S^6. A standard transversality argument shows that M_3 bounds.

REMARK 2. There is also an immersion theoretic proof of the theorem, worked out by Kirby and Freedman [1].

We conclude with a problem.

PROBLEM. (D. Ruberman) Modify some variant of Rohlin's proof to give a topological computation of the 4-dimensional oriented spin bordism group.

APPENDIX

For the convenience of the reader, here is an English translation of the French translation by L. Guillou and V. Sergiercu of Section 2 of Rohlin's article [4]:

THEOREM. M^4 <u>bounds if and only if</u> $\sigma(M^4) = 0$ . . .

[M^4 is an oriented, closed smooth 4-manifold of signature $\sigma(M^4)$.] This follows from:

LEMMA A. <u>For every</u> M^4 <u>there exists an integer</u> s <u>such that</u> $M^4 \sim s\mathbb{C}P^2$.

[$\sim$ denotes "is bordant to"]

LEMMA B. <u>If</u> $M^4 \sim N^4$, <u>then</u> $\sigma(M^4) = \sigma(N^4)$.

LEMMA C. $\sigma(s\mathbb{C}P^2) = s$.

<u>PROOF OF A.</u> One shows easily that $M^4 \sim M_1^4 \subset \mathbb{R}^7$. On M_1^4 one can find a normal vector field with isolated singularities of index ± 1. We seek $M_2^4 \sim M_1^4 + n\mathbb{C}P^2$, $M_2^4 \subset \mathbb{R}^7$, having a nonzero normal vector field. To achieve this, form the connected sum about each singularity with a $\mathbb{C}P^2$. Let L^7 be the complement of a tubular neighborhood of M_2^4 in S^7, and U^5 be the generator of $H_5(L^7, \partial L^7) \cong H_5(S^7, M_2^4) \cong H_4(M_2^4)$ determined by the orientation of M_2^4. Among the cycles representing U^5 one can find a manifold whose boundary

is bordant to $M_2^4 + m\mathbb{C}P^2$ for some integer m. Thus $M^4 \sim M_1^4 \sim M_2^4 + m\mathbb{C}P^2 - (m+n)\mathbb{C}P^2 \sim -(m+n)\mathbb{C}P^2$.

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