FIBRED KNOTS OF GENUS 2 FORMED BY PLUMBING HOPF BANDS

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ABSTRACT

Fibred knots of genus 2 have Conway polynomial $1+c_1z^2\pm z^4$. We show that the polynomials $1+4kz^2+z^4$ and $1+(4k+2)z^2-z^4$ cannot arise for a knot formed by plumbing Hopf bands. Further properties of the monodromy mod 2 in the case where c_1 is even shows that none of Burde's genus 2 fibred knots with c_1 even are formed by plumbing Hopf bands.

In the study of fibred knots and links the technique of plumbing two embedded surfaces $F_1, F_2 \subset S^3$ to get another embedded surface $F = F_1 \cup F_2$ plays a significant role. This follows the result [6] that, if two fibre surfaces F_1 and F_2 are plumbed, then the resulting $F = F_1 \cup F_2$ is also a fibre surface, that is, $S^3 - \partial F$ is fibred over S^1, with F forming one fibre.

Many examples of fibre surfaces can be built up from simpler ingredients in this way. Starting, for example, with one of the simplest fibre surfaces, the positive or negative Hopf band, that is, a closed unknotted ribbon with a single positive or negative full twist, and successively plumbing on further Hopf bands will generate quite a number of fibred knots of a given genus.

In this paper we investigate fibred knots of genus 2 which arise by plumbing Hopf bands, and give necessary, but not sufficient, conditions in terms of its Alexander polynomial for a fibred knot of genus 2 to arise in this way. These conditions enable us to provide examples of fibre surfaces which are not the plumbing of Hopf bands using, for example, Burde's sequence $K(c_1, \pm 1)$ of fibred knots (see Figure 1).

The example in Harer's paper [3] attributed to us, of a fibred knot which does not arise by plumbing Hopf bands, is $K(2, 1)$. In fact this knot is not excluded by our test on the Alexander polynomial, for it has the same polynomial as the sum of two trefoils; when Harer's paper was written, our results simply said that certain Alexander polynomials could not occur. However, if the sign of one of the Hopf bands is changed, the knot becomes $K(2, -1)$, which can be excluded by the test.

By a closer look at the Seifert form for a Hopf plumbing of genus 2 we have now shown the following.

THEOREM 5. *None of Burde's knots $K(c_1, \pm 1)$ are Hopf plumbings when c_1 is even.*

Harer's example is then justified, although it was not the knot originally envisaged by us.

This theorem and our Alexander polynomial condition both stem from an explicit investigation of the possible Seifert matrices for a knot given by Hopf plumbing. The

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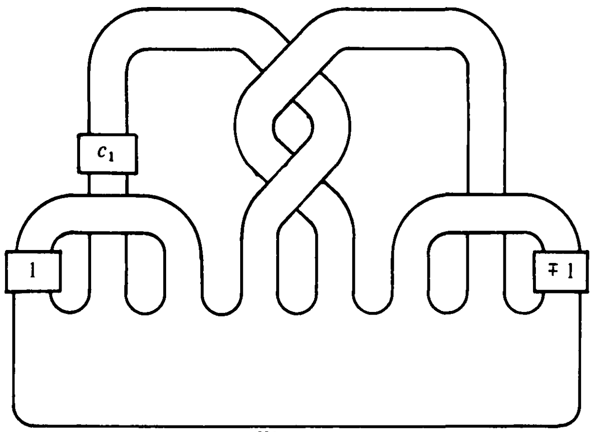

$K(c_1, \pm 1)$

FIG. 1

condition on the Alexander polynomial is most readily stated in terms of Conway's normalised form [2, 4], of the polynomial $\nabla_K(z)$ for a knot K. The polynomial $\nabla_K(z)$ can be recovered from the Alexander polynomial $\Delta_K(t)$, by first 'balancing' Δ_K, that is, multiplying by some power of t to write it in the form $a_0+\Sigma_{k=1}^n a_k(t^k+t^{-k})$, changing sign if necessary to ensure that $\Delta_K(1) = +1$, and finally writing it as a polynomial in $z = x-x^{-1}$ with $x^2 = t$. It can, however, be calculated directly from a Seifert matrix A for the knot K by $\nabla_K(z) = \det(xA-x^{-1}A^T)$, putting $z = x-x^{-1}$. For a knot K, $\nabla_K(z)$ is in fact a polynomial in z^2, since $z^2 = t+t^{-1}-2$. If K is fibred of genus 2 then $\nabla_K(z) = 1+c_1z^2 \pm z^4$, realised by $K(c_1, \pm 1)$ for any $c_1 \in \mathbb{Z}$. (For a discussion of Burde's knots in general, see [1, 5].)

THEOREM 3. *If a fibred knot K of genus* 2 *can be constructed by plumbing Hopf bands, then*

$$\nabla_K(z) \neq \begin{cases} 1+c_1z^2+z^4 & \textit{for } c_1 = 0 \bmod 4, \\ 1+c_1z^2-z^4 & \textit{for } c_1 = 2 \bmod 4. \end{cases}$$

Besides these exclusions there are further restrictions on the possible even values of c_1, depending on the values taken by certain integer quadratic forms. All odd values of c_1 can, however, be realised by plumbing knots.

In the paper quoted, [3], Harer proves that every fibre surface in S^3 results from a disc by a sequence of elementary changes:

(a) plumb on a Hopf band,
(b) de-plumb a Hopf band, that is, the inverse of (a),
(c) perform a Dehn twist about a suitable unknotted curve in the fibre.

He asks whether changes of either type (b) or (c) can be omitted, and any fibre surface realised using only the remaining two types. We shall conclude by showing how all the knots $K(c_1, \pm 1)$, which come from a disc by plumbing and twisting alone, can equally be generated simply by plumbing and de-plumbing.

2. *Hopf plumbing*

To plumb a Hopf band H to a fibre surface F_1, choose a square $P_1 \subset F_1$ with two opposite sides in ∂F_1 and then glue a similar square $P_2 \subset H$ to P_1 matching the sides in $P_1 \cap \partial F_1$ with those in P_2 which do not lie in ∂H, so that the whole of H lies on one side of F_1 in some neighbourhood of $P_1 \times I$.

A more usual equivalent procedure is to arrange F_1 so that P_1 is visible and then place H so as to overlay F_1, with P_1 and P_2 matching as before (see Figure 2). It is, however, useful not to have to move F_1 in trying to visualise the plumbing.

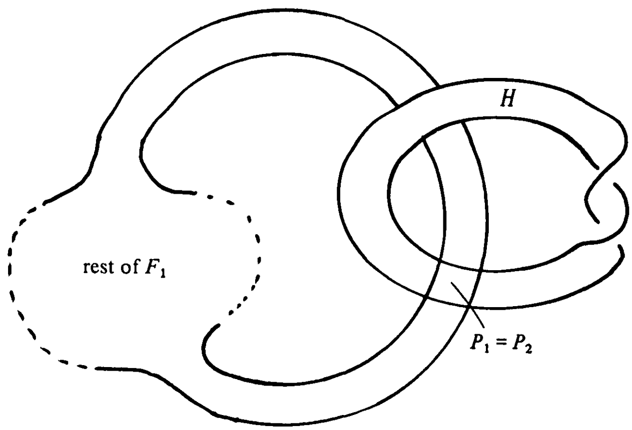

FIG. 2

A picture more like Figure 3 can also be helpful. The choice of P_1 corresponds simply to a neighbourhood of some arc a_1 in F_1 whose ends lie in ∂F_1. Near this arc we add a twisted band to the surface lying close to a_1 which, together with the neighbourhood of a_1 makes up the Hopf band. It does not matter which side of F_1 is used, for the monodromy of F_1 gives an isotopy carrying F_1 through S^3 to lie just to the other side of the band, if we so wish. Indeed, since the fibre of an *oriented* fibred link is determined up to isotopy by its boundary, we should not expect any difference to arise from the choice of side since the boundary of the plumbed surface is the same in either case.

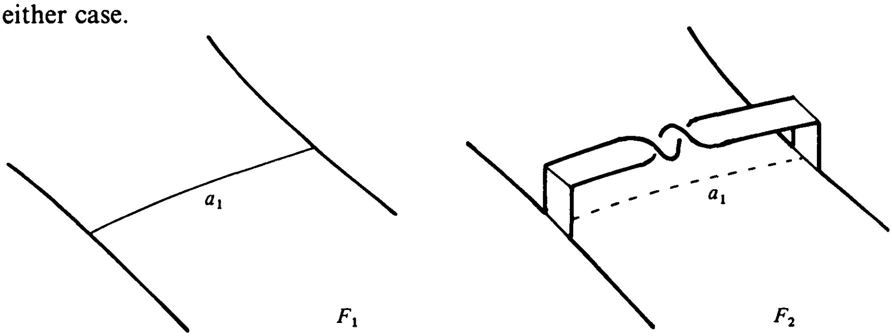

FIG. 3

A theorem of Stallings [6] shows that if a Hopf band is plumbed on to a fibre surface F_1 then the resulting surface F is also a fibre surface, so that the oriented link ∂F is fibred with F as one fibre. Now $r(F) = rk(\pi_1 F)$ increases by one for each Hopf band, that is, $r(F) = r(F_1)+1$ in the construction described.

DEFINITION. We say that F is a *Hopf plumbing* if it is constructed from D^2 by successively plumbing r Hopf bands, for some r. Then F is a fibre surface with $r(F) = r$. We shall say that the oriented link ∂F is given by Hopf plumbing.

3. *Genus* 2 *knots given by Hopf plumbing*

To study the case where $K = \partial F$ is a fibred knot and F, of genus 2, is a Hopf plumbing, we must consider, since $r(F) = 4$, all possible sequences of surfaces $F_0 = D^2$, F_1, F_2, F_3, $F_4 = F$, where F_{i+1} is given by plumbing a Hopf band on to F_i.

The surface F_{i+1} depends on F_i and on the choice of arc a_{i+1} in F_i to be used in the plumbing, as well as the sign of the Hopf band used. If a_{i+1} connects two points in the same component of ∂F_i then F_{i+1} will have one more boundary component than F_i, otherwise it will have one less. Since F_4 is to have one boundary component, the number of boundary components of $F_0, \ldots, F_4$ will form a sequence, either

(a) 1, 2, 1, 2, 1, or
(b) 1, 2, 3, 2, 1.

The choice of arc a_1 is automatic. We shall show that in case (b) there is a very limited choice for a_2 and a_3, and that any resulting F_3 could also be made by a plumbing of type (a). Bear in mind that a_{i+1} may be varied by isotopy in F_i, with the ends free to move in ∂F_i, without altering F_{i+1}. Indeed a_{i+1} can also be replaced by $h_i(a_{i+1})$, where $h_i : F_i \to F_i$ is the monodromy for F_i, since the isotopy of F_i in S^3 which realises h_i will carry a band determined by a_{i+1} to a band determined by $h_i(a_{i+1})$.

We start then with $F_1 = H^{\pm}$, one of the two Hopf bands. There is just one choice of a_2 joining the two components of ∂F_1. This arc (case (a)) gives F_2 as the fibre surface for a left- or right-handed trefoil, or figure-eight knot, according to the signs of the bands used.

If a_2 joins one component of ∂F_1 to itself (case (b)) then F_2 is the connected sum of two Hopf bands (Figure 4), and a_3, joining two components of ∂F_2, must be one of the three arcs a_3, a_3', a_3'' shown. In each case the resulting F_3 can be seen to arise also from one of the surfaces F_2 of case (a) by plumbing in a different order.

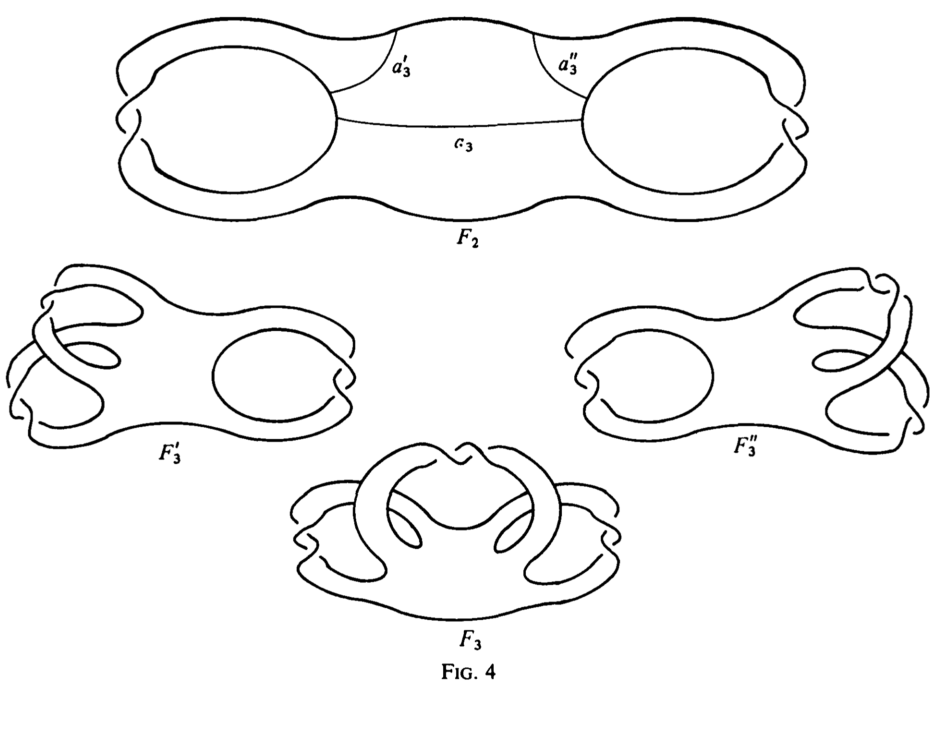

FIG. 4

We may then concentrate on case (a), and look at the surfaces $F_3 \cong T^2-2$ discs arising from the choices of a_3 in $F_2 \cong T^2-$disc, and finally at the choice of a_4 in F_3 joining the two boundary components which gives our possible selection of F_4.

THEOREM 1. *If K is a fibred knot given by Hopf plumbing then we can find an upper triangular Seifert matrix for K. If in addition K has Conway polynomial* $\nabla_K(z) = 1+c_1 z^2 \pm z^4$ *with* c_1 *even, then K has a Seifert matrix*

$$A = \begin{pmatrix} \pm 1 & 1 & 0 & m \\ 0 & \pm 1 & 0 & n \\ 0 & 0 & \pm 1 & 1 \\ 0 & 0 & 0 & \pm 1 \end{pmatrix}.$$

Proof. Since K is fibred and ∇_K has degree 4 the fibre must be some F_4 of genus 2. A Seifert matrix for F_4 can be found using a sequence of embedded curves x_1, x_2, x_3, x_4, consisting of the cores of the successive Hopf bands, as a basis for $H_1(F_4)$. The curve x_i is assumed to extend the arc a_i used in plumbing the i-th band along the core of the band. Let a_k have intersection number m_{ik} with x_i, $i < k$, in F_{k-1}. Then x_k will have linking number m_{ik} with x_i when it is pushed off F in one direction, and 0 when pushed off in the other direction ($i < k$). Its self-linking number will be ± 1 depending on the sign of the band. The Seifert matrix in this basis will be upper triangular, with entries ± 1 on the diagonal, m_{ik} above.

Having restricted to case (a) we may assume that $m_{12} = 1$, choosing the sign of x_2 as required. Since F_2 is a torus with a hole, the embedded arc a_3 will meet the generators x_1, x_2 of $H_1(F_2)$ p and q times respectively, where either p and q are coprime, or $p = q = 0$ and a_3 lies close to ∂F_2.

Write

$$A = \begin{pmatrix} \alpha & 1 & p & m \\ 0 & \beta & q & n \\ 0 & 0 & \gamma & s \\ 0 & 0 & 0 & \delta \end{pmatrix}$$

for the Seifert matrix of F_4. It must satisfy $\det(A-A^T) = 1$, since $A-A^T$ represents the non-singular intersection form on $H_1(F)$.

When $(p, q) \neq (0, 0)$ then p and q are not both even. We can assume in this case that p is even and q is odd, by using $h_2(a_3)$ or $h_2^2(a_3)$ if necessary. For the monodromy h_2 is the composite of a twist about x_1 and about x_2, and $h_2(a_3)$ then meets x_1, x_2 respectively p' and q' times with $q' = q \pm p, p' = p \pm q'$. In this case we can assume that

$$A = \begin{pmatrix} 1 & 1 & 0 & m \\ 0 & 1 & 1 & n \\ 0 & 0 & 1 & s \\ 0 & 0 & 0 & 1 \end{pmatrix} \bmod 2,$$

and $\det(A-A^T) = (m+s)^2 \bmod 2$, giving $(m, s) = (1, 0)$ or $(0, 1) \bmod 2$. Calculation of the Conway polynomial mod 2 gives $1+z^2+z^4$ in either case.

It follows that if c_1 is even then $p = q = 0$. In this case $\det(A-A^T) = s^2 = 1$, so by choice of orientation of x_4 we have $s = 1$. This completes the proof of Theorem 1.

In this case x_3 is parallel to one boundary circle of F_3, so the arc a_4 joining the two boundary circles of F_3 will clearly meet x_3 once. There is no restriction on m and

n here; in a torus with two holes an arc joining the boundary components may meet x_1 and x_2 any number of times, so all matrices

$$A = \begin{pmatrix} \alpha & 1 & 0 & m \\ 0 & \beta & 0 & n \\ 0 & 0 & \gamma & 1 \\ 0 & 0 & 0 & \delta \end{pmatrix}, \qquad \alpha\beta\gamma\delta = \pm 1,$$

will occur for some Hopf plumbing.

THEOREM 2. *If K has Seifert matrix*

$$A = \begin{pmatrix} \alpha & 1 & 0 & m \\ 0 & \beta & 0 & n \\ 0 & 0 & \gamma & 1 \\ 0 & 0 & 0 & \delta \end{pmatrix}$$

then $\nabla_K(z) = 1+\{\alpha\beta+\gamma\delta+\gamma(\alpha n^2+\beta m^2-mn)\}z^2+\alpha\beta\gamma\delta z^4$.

Proof. This follows by direct calculation from $\nabla_K(z) = \det(xA-x^{-1}A^T)$, putting $z = x-x^{-1}$.

Our main condition on the Alexander polynomial of a Hopf plumbing now follows readily.

THEOREM 3. *If a fibred knot K, of genus 2, is given by Hopf plumbing, then*

$$\nabla_K(z) \neq \begin{cases} 1+c_1 z^2+z^4 & \text{with } c_1 = 0 \bmod 4 \\ 1+c_1 z^2-z^4 & \text{with } c_1 = 2 \bmod 4. \end{cases}$$

Proof. For a genus 2 Hopf plumbing K, $\nabla_K(z)$ is given by Theorem 2 when c_1 is even. In that formula, c_1 is even if and only if the quadratic expression $\alpha n^2+\beta m^2-mn$ is even. This in turn requires that m and n are both even, so that

$$\nabla_K(z) = 1+(\alpha\beta+\gamma\delta)z^2+\alpha\beta\gamma\delta z^4 \bmod 4.$$

If $\alpha\beta\gamma\delta = 1$ then $\alpha\beta+\gamma\delta = \pm 2$, otherwise $\alpha\beta\gamma\delta = -1$ and $\alpha\beta+\gamma\delta = 0$, so the only possible polynomials with even c_1 are $1+2z^2+z^4$ and $1-z^4$, modulo 4.

The remark that m and n are both even can be expressed as follows.

THEOREM 4. *If K is given by Hopf plumbing and has* $\nabla_K(z) = 1+c_1 z^2 \pm z^4$ *with* c_1 *even then K has a Seifert matrix A congruent to*

$$\begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \bmod 2.$$

COROLLARY. *The monodromy matrix* $H = A(A^T)^{-1}$ *of such a knot K then satisfies* $H^2+H+I = 0 \bmod 2$*; in other words the minimal polynomial* mod 2 *of the monodromy has lower degree than its characteristic polynomial.*

4. *Fibred knots not given by Hopf plumbing*

The conditions of Theorem 3 allow us to exhibit many fibred knots which are not given by Hopf plumbing. It is also possible by further investigating the quadratic form in Theorem 2 to give more restrictions on the even values of c_1 which can occur.

The corollary to Theorem 4 suggests that in fact Hopf plumbings with even c_1 are quite rare. Indeed we can use it to prove the following.

THEOREM 5. *None of Burde's fibred knots $K(c_1, \pm 1)$ with c_1 even are given by Hopf plumbing.*

Proof. Choose a Seifert matrix B for $K(c_1, \pm 1)$, using the cores of the bands in the diagram in Figure 1 as a basis. Then

$$B = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \bmod 2$$

when c_1 is even. Hence the monodromy matrix

$$L = B(B^T)^{-1} = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 \end{pmatrix} \bmod 2.$$

If the knot is a Hopf plumbing then L must be conjugate in $GL(4, \mathbb{Z})$ to a matrix H as in the corollary to Theorem 4, and so $L^2+L+I = 0 \bmod 2$. Direct calculation shows that

$$L^2+L+I = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix} \bmod 2.$$

REMARK. Similar calculations show that the (2, 1) cables about the trefoil and figure-eight knots, which are fibred satellite knots with Conway polynomials $1+4z^2+z^4$ and $1-4z^2-z^4$ respectively, cannot be Hopf plumbings. The first can be excluded immediately, by Theorem 3, and the second by its monodromy as in Theorem 5.

We should also note that all polynomials $1+(2n+1)z^2 \pm z^4$ *can* occur from Hopf plumbings. Start with the surface F_3, with x_1, x_2, x_3 as shown in Figure 5, and find an arc a_4 having intersection numbers $0, n, n+1$ respectively with x_1, x_2, x_3.

Plumb on a $\pm$ Hopf band along a_4 to get F_4 with Seifert matrix

$$\begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & n \\ 0 & 0 & -1 & n+1 \\ 0 & 0 & 0 & \pm 1 \end{pmatrix}.$$

The resulting knot has $\nabla(z) = 1+(2n+1)z^2 \mp z^4$.

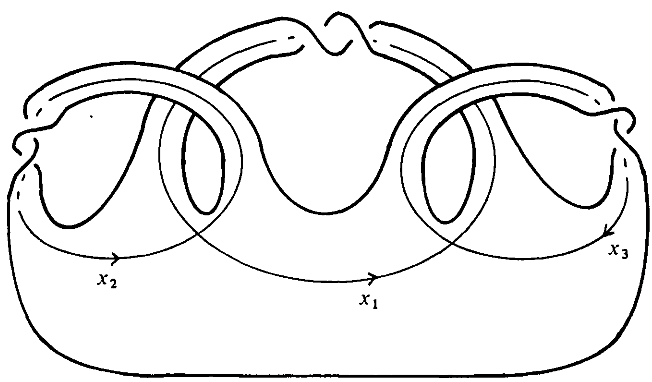

FIG. 5

5. *Stable Hopf plumbing*

DEFINITION. Two fibre surfaces F and F' are *Hopf equivalent* ($F \sim F'$) if F is obtained from F' by plumbing and de-plumbing Hopf bands. Observe that if F and F' are both obtained from some G by plumbing a Hopf band, along arcs a and a' respectively, then the surfaces given by plumbing a band on F along $a' \subset G \subset F$ and on F' along a will be isotopic; we can choose to place the bands on either side of G without affecting the resulting surface up to isotopy. It follows that $F \sim F'$ if and only if there is some surface E given by Hopf plumbing from both F and F'. If $F \sim$ disc, we say F is a *stable Hopf plumbing*, and the oriented link ∂F is given by stable Hopf plumbing.

THEOREM 6. *All of Burde's genus 2 fibred knots $K(G, \pm 1)$ are given by stable Hopf plumbing.*

Proof. The fibre of $K(c_1, \pm 1)$ is clearly Hopf equivalent to the surface $L_{c_1} \cong M_{c_1}$ shown in Figure 6. Since M_1 is a Hopf plumbing (it is the connected sum of two Hopf bands) the theorem follows from repeated application of the following lemma, which shows that $M_k \sim M_{k-1}$.

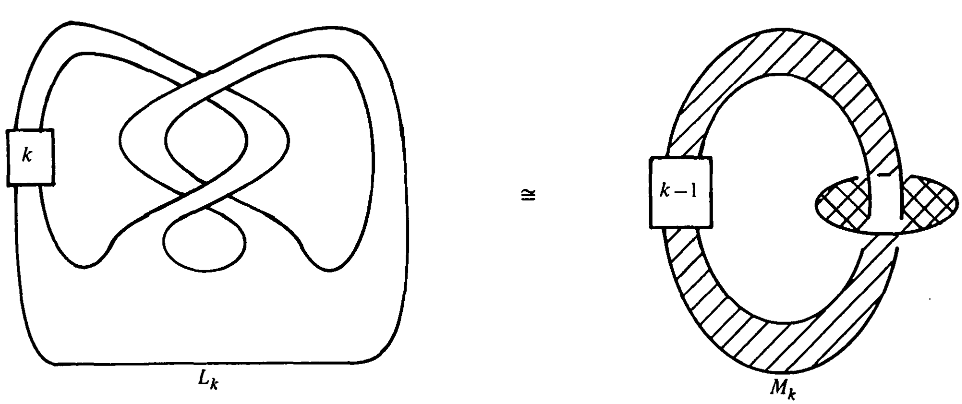

FIG. 6

LEMMA. *If fibre surfaces F and F′ are related by changing H to H′* (Figure 7), *where F and F′ are assumed to meet some ball B^3 in H and H′ respectively and agree outside B^3, then $F \sim F'$.*

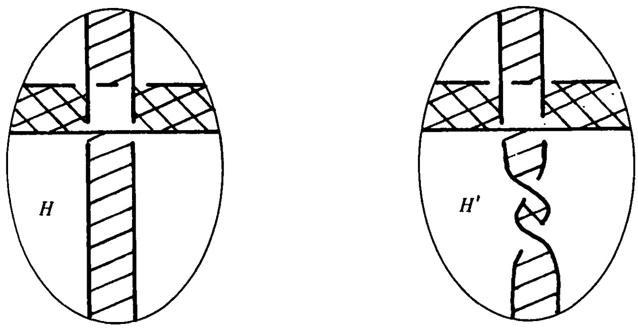

FIG. 7

Proof. Plumb on a Hopf band along a in F as shown. With suitable choice of sign the new part of the ribbon (lying close to one side of F) appears untwisted in the diagram because of the twist in F along a. Isotop the surface, within B^3, until it appears as H' with a band plumbed along c' (see Figure 8).

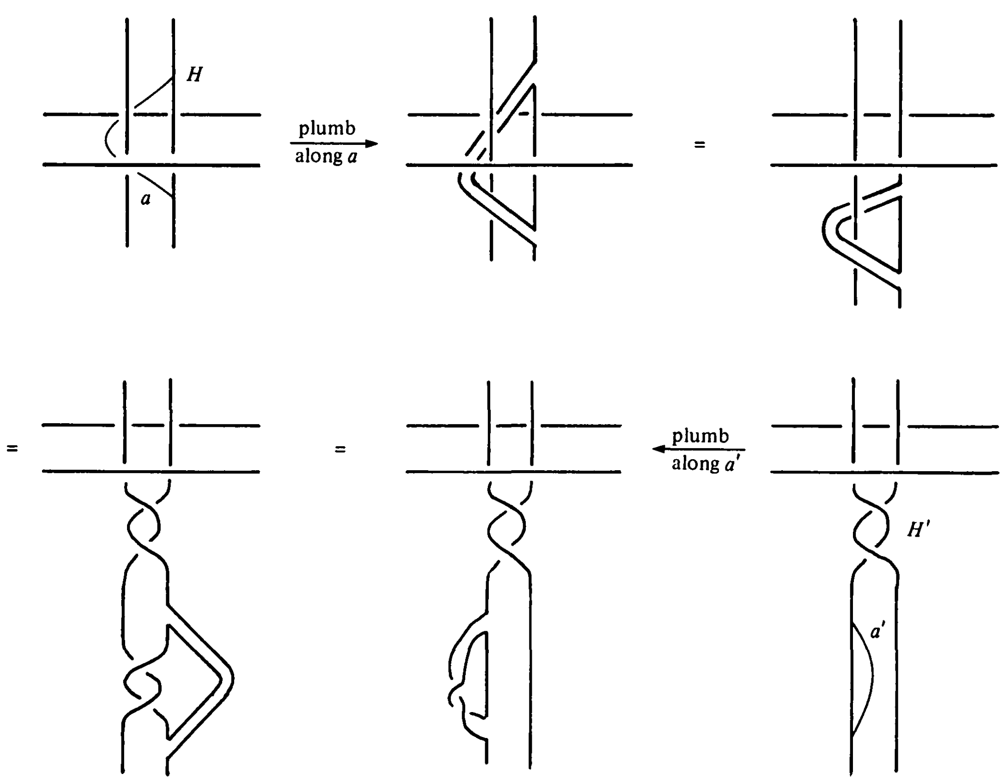

FIG. 8

A natural question to ask is whether Burde's higher genus knots are all given by stable Hopf plumbing. One might suspect so, as they come from plumbing a sequence of surfaces L_k [5] each of which is a stable Hopf plumbing by the lemma. However, if fibre surfaces T and T' are obtained by plumbing F and F' respectively to a surface S, with $F \sim F'$ it is not clear that $T \sim T'$. One can certainly find $U \sim T$ and $U' \sim T'$ each given by plumbing a surface E to S, but there is no obvious reason why different ways of plumbing two surfaces, E and S, should result in Hopf equivalent surfaces.

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