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4-MANIFOLDS WITH LARGE SYMMETRY GROUPS

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THE study of manifolds with continuous symmetry groups has always been a fruitful avenue of research, both for its inherent beauty and for its contribution to the general study of manifolds. In dimension 3, for example, the S^1-manifolds studied by Seifert have turned out to be basic building blocks in the structure theorems of Jaco, Shalen, Johannson and Thurston. One might reasonably hope for a similar phenomenon in dimension 4.

This paper gives a diffeomorphism classification of all closed oriented 4-manifolds which support an effective action of some compact non-abelian Lie group (Theorem 3.7). They fall into the following classes:

(1) S^4 or $\pm \mathbb{C}P^2$
(2) connected sums of copies of $S^1 \times S^3$ and $S^1 \times P^3$
(3) $(SU(2)/H)$-bundles over S^1 (H a finite subgroup of $SU(2)$)
(4) S^2-bundles over surfaces
(5) certain quotients of S^2-bundles over surfaces by involutions.

The restriction to *non-abelian* Lie groups avoids the difficult problem of classifying 4-manifolds with S^1 or T^2-actions [3, 4, 9–11]. (It is well known that any 4-manifold with an effective T^n-action, $n \geqq 3$, is T^4 or $S^1 \times L$, where L is a lens space.) This paper thus reduces the general classification problem for closed, orientable 4-manifolds with a compact Lie group G of symmetries to the cases $G = S^1$ or T^2.

In §1 we show that one need only consider actions of $SU(2)$ or $SO(3)$, and the well known subgroup structure of these two groups is recalled. A complete equivariant classification of $SU(2)$ and $SO(3)$-actions on 4-manifolds (including the non-orientable case) is given in §2. The codimension 1 case is taken largely from the second author's thesis [12]. The final section gives the topological classification in the orientable case. This leaves open the non-orientable case, where the situation seems quite interesting and more intricate (cf. [13] for the codimension 1 case). We plan to take this up in a sequel to this paper.

We shall work in the smooth category. B^n will denote the n-ball, S^n the n-sphere, and P^n the real projective n-space. The reader is referred to Bredon's book [1] for the basic definitions and theorems of transformation groups.

§1. SUBGROUPS

LEMMA 1.1. *If a connected 4-manifold M supports an effective action of a compact non-abelian Lie group G, then it supports an effective action of $SU(2)$ or $SO(3)$.*

Proof. It suffices to show that G contains $SU(2)$ or $SO(3)$ as a subgroup. By the hypothesis

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on G, $\dim(G) \leqq 10$ [2, p. 239]. By the classification of compact connected Lie groups [1, p. 30], the connected component of the identity in G is isomorphic to $(A \times S)/H$ where A is abelian, S ($\neq 1$ since G is non-abelian) is simply-connected and semisimple, and H is a subgroup of the center of $A \times S$. In particular each simple factor of S is either one of the simply-connected Lie groups $\mathrm{Sp}(n)$, $SU(n+1)$, or $\mathrm{Spin}(n+2)$ $(n \geqq 1)$, or one of the five exceptional groups. The exceptional groups cannot occur, however, as they have dimension $\geqq 14$ while $\dim(S) \leqq \dim(G) \leqq 10$. Since $\mathrm{Sp}(1) = SU(2) = \mathrm{Spin}(3)$, it follows that $SU(2) \subset S$. Now the center Z of $SU(2)$ is cyclic of order 2, with $SU(2)/Z = SO(3)$. Thus $SU(2)$ or $SO(3)$ is a subgroup of $(A \times S)/H \subset G$. □

To study the actions of a Lie group one must know a lot about its closed subgroups. For $SO(3)$ and $SU(2)$ these are well understood.

First consider the rotation group $SO(3)$. To fix notation, let C_n denote the (unique) cyclic subgroup of $SO(3)$ of order $n \geqq 1$ whose elements fix the z-axis. Extending C_n by x^2, where $x \in SO(3)$ is rotation by $\pi/2$ radians about the x-axis, yields a dihedral subgroup D_{2n} of order $2n$. Note that $C_1 = 1$ and D_2 is conjugate to C_2. Let T_{12}, O_{24}, and I_{60} be the subgroups containing D_4 which are isomorphic respectively to the symmetry groups of the tetrahedron, octahedron, and icosahedron. Finally let $SO(2)$ and $O(2)$ be the subgroups containing C_n and isomorphic respectively to the circle group and the orthogonal group of the plane. It is known that each closed subgroup of $SO(3)$ is conjugate to exactly one of the subgroups

$$C_n\ (n \geqq 1),\ D_{2n}\ (n \geqq 2),\ T_{12},\ O_{24},\ I_{60},\ SO(2),\ O(2),\text{ or } SO(3)$$

(see [16]).

The closed subgroups of $SU(2)$ are described as follows: Let $p: SU(2) \to SO(3)$ be the universal (2-fold) covering homomorphism. For each closed subgroup H of $SO(3)$, set

$$H^* = p^{-1}H.$$

Then the closed subgroups of $SU(2)$ are exactly the subgroups H^* together with the odd cyclic subgroups. As with $SO(3)$, isomorphic subgroups are conjugate. Note that the subgroups $SO(2)^*$ and $O(2)^*$ are isomorphic respectively to $SO(2)$ and $O(2)$. There is a unique cyclic subgroup of $SO(2)^*$ of any order n, denoted also by C_n. In particular $C_{2n} = C_n^* \subset SU(2)$.

The study of effective actions of a specific compact Lie group G on connected n-manifolds usually begins with a list of the possible *principal isotropy types* (H) [1, p. 179]. H must satisfy the two conditions

(1) $\dim(G/H) \leqq n$ $(n - \dim(G/H)$ is called the *codimension* of the action)
(2) the intersection of all the conjugates of H is trivial (to guarantee effectiveness).

For $SO(3)$-actions on 4-manifolds any proper closed subgroup satisfies these conditions, while for $SU(2)$-actions only the odd cyclic ones do. (The other subgroups of $SU(2)$ contain the center of $SU(2)$ and so condition (2) is violated). This yields

LEMMA 1.2. *Effective actions of $SU(2)$ on connected 4-manifolds are all of codimension 1 with principal isotropy C_n, n odd. Those of $SO(3)$ are of codimension 1 or 2. The principal isotropy is finite in the first case, and $SO(2)$ or $O(2)$ in the second.* □

§2. EQUIVARIANT CLASSIFICATION

In this section we give an equivariant classification of effective actions of $SU(2)$ and $SO(3)$ on closed connected (not necessarily orientable) 4-manifolds. The codimension 1 classification, originally given by the second author [12, 13], is based on the work of Mostert [6] and Neumann [7]. The codimension 2 classification builds on the diffeomorphism classification of S^2-bundles over surfaces given in [5]. It was obtained previously in the orientable case by Orlik [8].

We begin by defining the prototypes for codimension 1 actions of a compact connected Lie group G, found by Mostert.

Let H be a closed subgroup of G and n be an element of the normalizer $N(H)$ of H in G. Denote by

$$r_n: G/H \to G/H$$

the automorphism of the left G-space G/H given by right multiplication by n, that is $r_n(gH) = gHn = gnH$. Following the notation of [13] let

$$S_G(H)_n = \frac{(G/H) \times I}{(gH, 0) \sim (gnH, 1)}$$

denote the mapping torus of r_n, equipped with the natural left G-action ($g[g'H, t] = [gg'H, t]$, where $[gH, t]$ is the equivalence class of (gH, t)).

If K is a closed subgroup of G containing H, then (H, K) is called an *admissible pair* if K/H is diffeomorphic to a sphere. For admissible pairs (H, K_i) ($i = 0, 1$) set

$$I_G(H, K_0, K_1) = \frac{(G/K_0) \cup ((G/H) \times I) \cup (G/K_1)}{gK_0 \sim (gH, 0), (gH, 1) \sim gK_1}$$

with the natural left G-action (as above). $I_G(H, K_0, K_1)$ is the union of the mapping cylinders Z_i of the natural projections $G/H \to G/K_i$ along their common boundary G/H. The Z_i are disc bundles over G/K_i, since the (H, K_i) are admissible, and so $I_G(H, K_0, K_1)$ is a manifold. As a notational convenience set

$$I_G(H, K_0, K_1)_n = I_G(H, K_0, nK_1n^{-1})$$

for any n in $N(H)$. This may be viewed as the union of the Z_i with boundaries identified by r_n.

Mostert [6] showed that every closed connected manifold with an effective codimension 1 G-action is equivariantly diffeomorphic to some $S_G(H)_n$ or $I_G(H, K_0, K_1)_n$ (see also [7], [1, p. 206]). Following [13] we shall call these *Mostert manifolds*. Neumann [7] observed that $S_G(H)_n$ and $S_G(H)_{n'}$ are equivariantly diffeomorphic if and only if n and n' are in the same component of the group

$$\Gamma(H) = N(H)/H.$$

Similarly $I_G(H, K_0, K_1)_n$ are classified (for fixed H, K_0 and K_1) by the components of the double coset space

$$\Gamma(H, K_0, K_1) = N_0 \backslash N(H)/N_1$$

where $N_i = N(H) \cap N(K_i)$ ($i = 0, 1$). An appropriate analysis of the closed subgroups of G would thus yield a classification of the effective codimension 1 G-actions on manifolds.

For $G = SU(2)$ and $SO(3)$ acting on 4-manifolds, this analysis has been carried out in [12]. We tabulate the results below.

Table 1 gives one representative H from each potential principal isotropy type of codimension 1 G-action (Lemma 1.2) and the corresponding normalizer $N(H)$.

It follows from this table that $\Gamma(H) = N(H)/H$ is connected except when $H = C_n$ ($n \geqq 2$, $G = SU(2)$ or $SO(3)$), D_4 or T_{12}. (This is evident for $H = D_{2n}$ ($n \geqq 3$) since the rotation x^2 (see §1) lies in D_{2n}, and in the remaining cases $N(H)$ is either connected or equal to H.) Table 2 gives a set $n(H)$ of elements of $N(H)$, one from each component of $\Gamma(H)$. Here (and in the next two tables) x, y and z are respectively rotations by $\pi/2$ radians about the x, y and z axes (or lifts to $SU(2)$). S_3 is the symmetric group on 3 letters.

Table 1. Principal isotropy types (H)

G	H	$N(H)$
$SU(2)$	1	$SU(2)$
	C_n (odd $n \geqq 3$)	$O(2)^*$
$SO(3)$	1	$SO(3)$
	$C_n (n \geqq 2)$	$O(2)$
	D_4	O_{24}
	$D_{2n} (n \geqq 3)$	$O(2)$
	T_{12}	O_{24}
	O_{24}	O_{24}
	I_{60}	I_{60}

Table 2. Normal representatives $n(H)$

H	$\Gamma(H)$	$n(H)$
C_n $(n \geqq 2)$	$O(2)$	$1, x^2$
D_4	S_3	$1, x, y, z, xy, yx$
T_{12}	Z_2	$1, y$
all remaining cases		1

Table 3 gives all possible admissible pairs (H, K) (up to pairwise conjugacy) with H taken from Table 1.

Table 3. Admissible pairs (H, K)

G	H	K
$SU(2)$	1	$SU(2)$
	C_n (odd $n \geqq 1$)	C_{2n}
		$SO(2)^*$
$SO(3)$	C_n $(n \geqq 1)$	C_{2n}
		D_{2n} $(n \neq 1)$
		$SO(2)$
	C_2	$xO(2)x^{-1}$ †
	D_{2n} $(n \geqq 2)$	D_{4n}
		$O(2)$
	T_{12}	O_{24}

† This case (conjugate to $(D_2, O(2))$) was omitted in [12].

Observe that $N(SO(2)) = O(2)$, whence $N(SO(2)^*) = O(2)^*$. It follows from Tables 1 and 3 that $\Gamma(H, K_0, K_1)$ is connected unless $H = D_4$. Table 4 gives a set $n(H, K_0, K_1)$ of elements of $N(H)$, one from each component of $\Gamma(H, K_0, K_1)$.

Table 4. Normal representatives $n(H, K_0, K_1)$

H	K_0	K_1	$\Gamma(H, K_0, K_1)$	$n(H, K_0, K_1)$
D_4	D_8	D_8	$D_8 \backslash O_{24} / D_8$	$1, x$
		$O(2)$		$1, x$
	$O(2)$	$O(2)$		$1, x$
all remaining cases				1

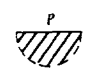
The discussion above yields the following codimension 1 classification theorem:

THEOREM 2.1. *Let M be a closed, connected 4-manifold with an effective codimension 1 G-action, $G = SU(2)$ or $SO(3)$. Then M is equivariantly diffeomorphic to exactly one of the Mostert manifolds $S_G(H)_n$ (where H is chosen from Table 1 and $n \in n(H)$ from Table 2) or $I_G(H, K_0, K_1)_n$ (where (H, K_i) are chosen from Table 3 and $n \in n(H, K_0, K_1)$ from Table 4).* □

We turn now to codimension 2 actions. Let M be a closed connected 4-manifold with an effective codimension 2 $SO(3)$-action. Let F denote the orbit space, H the principal isotropy, and X the complement in M of an open tubular neighborhood of the non-principal orbits of the action. It is well known that F is a compact connected surface [1, p. 186] (this can be seen directly from what follows) and that X fibers over its orbit space $E \subset F$ with fiber G/H and structure group $N(H)/H$ [1, p. 182].

By Lemma 1.2, $H = SO(2)$ or $O(2)$ with principal orbits S^2 or P^2, respectively. The possible non-principal orbits may be determined using the slice theorem. These are tabulated below, along with a corresponding isotropy subgroup and slice representation (which must be non-trivial [1, p. 181]), and the orbit space of an open tubular neighborhood of the orbit.

Table 5. Non-principal orbits

Orbit	Isotropy	Slice representation	Local orbit space
P^2	$O(2)$	$O(2) \to O(2)$ with image generated by a reflection	
		a rotation of order 2	
fixed point	$SO(3)$	standard inclusion $SO(3) \to O(4)$	

The points marked P and the point marked $\times$ correspond to P^2 orbits. Those marked F are fixed points. The remaining orbits are principal S^2 orbits.

If $H = O(2)$, then all orbits are principal. Thus F is closed and $M = X = F \times P^2$ (since $N(O(2)) = O(2)$).

If $H = SO(2)$ (assumed henceforth), then F may have boundary. Let f and p denote the number of components of ∂F consisting of fixed points and P^2 orbits, respectively, and write q for the number of isolated P^2 orbits. A typical orbit space is shown in Fig. 1 with $f = 1, p = 2$, $q = 3$.

Now X is the total space of an S^2-bundle ξ over $E \cong F - (q$ open discs) with structure group $\mathbb{Z}_2$ $(= O(2)/SO(2))$. The boundary ∂X of X is the union of $f + p + q$ S^2-bundles over S^1.

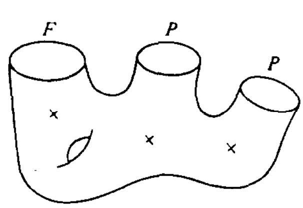


Fig. 1.

Each one corresponding to an isolated P^2 orbit must be the twisted bundle $S^1 \tilde{\times} S^2$ (by the slice theorem). Denote the number of twisted bundles corresponding to circles of fixed points by f'. Similarly define p'. Then

$$r = f' + p' + q$$

is the number of non-orientable components of ∂X. Observe that r is even. (To see this, let C be a properly embedded 1-manifold in E with a minimal number of components which is Poincaré dual to the first Stiefel–Whitney class $w_1 \xi$. Then r is the number of components of ∂E which C touches, cf. the structure lemma in [5].) Finally define

$$s = 0, 1, 2 \text{ or } \infty$$

as follows: if C is empty ($w_1 \xi = 0$) or ∂C is nonempty, set $s = 0$. If $w_1 \xi = w_1 E$, the first tangential Stiefel–Whitney class of E, set $s = \infty$. Otherwise, set $s = 1$ or 2 according to whether C is orientation reversing or preserving. It follows from [5, Theorem 2] that X is determined up to $SO(3)$-equivariant diffeomorphism by r and s.

It is now straightforward to show that

$$F,\ s,\ (f, f'),\ (p, p') \text{ and } q$$

form a complete set of invariants for the $SO(3)$-action on M. They satisfy

(1) F is a compact surface

(2) f, f', p, p' and q are non-negative integers with $f' \leqq f$, $p' \leqq p$, $f + p = rk(H_0 \partial F)$, and $r = f' + p' + q$ even.

(3) If $r > 0$ then $s = 0$. Otherwise s may be 0, 1, 2 or ∞, according to the following table

F	Genus (f)	s
orientable	0	∞
	>0	2, ∞
non-orientable	1	0, ∞
	2	0, 1, ∞
	>2	0, 1, 2, ∞

Conversely, one may construct an $SO(3)$-manifold

$$F(s, (f, f'), (p, p'), q)$$

with any prescribed values of F, s, f, f', p, p' and q satisfying (1)–(3). This yields

THEOREM 2.2. *Let M be a closed connected 4-manifold with an effective codimension 2 $SO(3)$-action. Then M is equivariantly diffeomorphic to a trivial P^2-bundle over a closed surface (with $SO(3)$ acting by rotation in the fiber), or to one of the $SO(3)$-manifolds $F(s, (f, f'), (p, p'), q)$ defined above.* □

The following orientability criterion will be used in the next section, where we give the topological classification in the orientable case.

LEMMA 2.3. (1) *$F(s, (f, f'), (p, p'), q)$ is orientable if and only if $s = \infty$ (and so f', p' and $q = 0$).*

(2) *Let H be a finite subgroup of a compact connected Lie group G. Then $S_G(H)_n$ is always orientable, and $I_G(H, K_0, K_1)_n$ is orientable if and only if K_0 and K_1 are infinite.*

Proof. (1) is straightforward from the definition.

To prove (2), observe that G is orientable and right multiplication r_g by any element g of G is orientation preserving. Since H is finite, it follows that G/H is orientable and the automorphism r_n of G/H is orientation preserving. Thus $S_G(H)_n$ is orientable.

Now recall that $I_G(H, K_0, K_1)_n$ is a union of the two mapping cylinders $Z_i (i = 0, 1)$ of the natural projections $G/H \to G/K_i$. If K_i is infinite, then G/K_i has codimension $\geqq 2$ in Z_i and so Z_i is orientable (since $G/H = \partial Z_i$ is). If K_i is finite, then Z_i is an I-bundle with orientable base G/K_i and connected boundary G/H, and so Z_i is nonorientable. Thus $I_G(H, K_0, K_1)_n$ is orientable if and only if the K_i are both infinite. □

For simplicity, we will denote the oriented $SO(3)$-manifold $F(\infty, (f, 0), (p, 0), 0)$ by

$$F(f)$$

in the sequel ($p = rk(H_0 \partial F) - f$).

Combining Lemma 2.3 with the classification theorems 2.1 and 2.2, we have

THEOREM 2.4. *Let M be a closed, connected oriented 4-manifold with an effective G-action, $G = SO(3)$ or $SU(2)$. Then M is equivariantly diffeomorphic to exactly one of the following G-manifolds:*

(1) *$S_G(H)_n$, with H taken from Table 1 and $n \in n(H)$ from Table 2.*

(2) *$I_G(H, K_0, K_1)$, with H, K_0 and K_1 taken from the following table*

G	H	K_0	K_1
$SU(2)$	1	$SO(2)^*$	$SU(2)$
		$SU(2)$	$SU(2)$
	C_n (odd $n \geqq 1$)	$SO(2)^*$	$SO(2)^*$
$SO(3)$	C_2	$SO(2)$	$O(2)'$
	D_4	$O(2)$	$O(2)'$
	C_n ($n \geqq 1$)	$SO(2)$	$SO(2)$
	D_{2n} ($n \geqq 1$)	$O(2)$	$O(2)$

where $O(2)' \equiv xO(2)x^{-1}$.

(3) $F(f)$, with F a compact surface and $0 \leqq f \leqq rk(H_0 \partial F)$ ($G = SO(3)$).

§3. TOPOLOGICAL CLASSIFICATION

The classification is given in Theorem 3.7. We first establish notation and give some preliminary results.

Let F be a compact surface. Write

$$M(F)$$

for the (unique) oriented spin 4-manifold which fibers over F with fiber S^2. In particular, $M(F) = F \times S^2$ if F is orientable. If F is closed then there is one other oriented 4-manifold

$$N(F)$$

underlying an S^2-bundle over F. It is obtained from $M(F)$ by twisting along an S^2 fiber (i.e. cutting along the boundary $S^1 \times S^2$ of a tubular neighborhood $B^2 \times S^2$ of the fiber and then reidentifying by the diffeomorphism of $S^1 \times S^2$ coming from the non-trivial element of $\pi_1 SO(3)$). $N(F)$ is not a spin manifold.

The following result is immediate from the homotopy sequence of a fibration.

LEMMA 3.1. (1) *$\pi_1 M(F) = \pi_1 F = \pi_1 N(F)$. Thus for closed surfaces F, the manifolds $M(F)$ (respectively $N(F)$) are classified up to diffeomorphism by F.*

(2) *$\pi_2 M(F)$ and $\pi_2 N(F)$ are both non-zero.* □

If F has boundary, define

$$P(F)$$

to be the manifold obtained from $M(F)$ by identifying antipodal points on each fiber of $\partial M(F)$. In particular, $P(F)$ is diffeomorphic to the $SO(3)$-manifold $F(0)$ (see §2).

LEMMA 3.2. (1) *If $F \neq B^2$ then $\pi_1 P(F)$ is infinite with non-trivial torsion.*

(2) *The manifolds $P(F)$ (for bounded surfaces F) are classified up to diffeomorphism by F.*

(3) *$\pi_2 P(F) \neq 0$.*

Remark 3.3. $P(B^2) = M(P^2)$. Indeed, there is a bundle projection $P(B^2) \to P^2$ induced by the projection $M(B^2) \to S^2$ to a fiber. One easily checks that $P(B^2)$ is spin.

Proof of 3.2. A Mayer–Vietoris argument shows that

$$H_1 P(F) = H_1 F \oplus \mathbb{Z}_2^{p(F)}$$

where $p(F)$ is the number of components of ∂F. If $F \neq B^2$ then $H_1 F$ is infinite, and so $\pi_1 P(F)$ is infinite. To see that $\pi_1 P(F)$ has torsion, observe that there is a map $P(F) \to P^2$ (constructed as in the remark above) which is a homeomorphism on any P^2 fiber P (the quotient of an S^2 fiber in $\partial M(F)$) in $P(F)$. Thus $\pi_1 P = \mathbb{Z}_2 \subset \pi_1 P(F)$. This proves (1).

Set $r(F) = rk(H_1 F)$. By the computation above, $r(F)$ and $p(F)$ are topological invariants of $P(F)$. To prove (2), we may assume that $F \neq B^2$, as $r(F) = 0$ if and only if $F = B^2$. We then have $r(F) \geqq p(F) - 1 \geqq 0$. Let r and p be nonnegative integers satisfying

$$r \geqq p - 1 \geqq 0.$$

If $r - (p-1)$ is odd or zero, then there is a unique F with $r(F) = r$ and $p(F) = p$, and so $P(F)$ is determined by F. If $r - (p-1) = 2n$ ($n > 0$), then there are exactly two surfaces F_i ($i = 0, 1$) with $r(F_i) = r$ and $p(F_i) = p$. In particular

$$F_0 = nT^2 \# pB^2$$
$$F_1 = nK^2 \# pB^2$$

where T^2 and K^2 are respectively the torus and the Klein bottle, and nM denotes the connected sum of n copies of M. We must show that $P(F_0) \neq P(F_1)$.

Observe that any 3-fold cyclic cover E_i of F_i induces a cover $P(E_i)$ of $P(F_i)$. The collection C_i of covering spaces of $P(F_i)$ which arise in this way correspond exactly to homomorphisms

$$h: H_1 P(F_i) \to \mathbb{Z}_3$$

with $\operatorname{tor}(H_1 P(F_i)) \subset \ker(h)$. Thus C_i is an invariant of $P(F_i)$. One easily shows that F_1 has a 3-fold cyclic cover with $3p$-2 boundary components, while F_0 does not. It follows that $C_0 \neq C_1$. Thus $P(F_0) \neq P(F_1)$. This proves (2).

To prove (3), observe that there is a 2-fold cover $M(DF) \to P(F)$, where

$$DF = F \underset{\partial}{\cup} -F$$

is the double of F, as indicated in Fig. 2. The covering transformation in $M(DF)$ is the

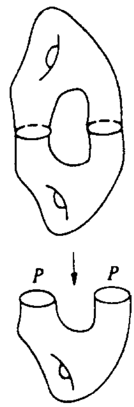


Fig. 2.

composition of reflection through ∂F with the antipodal map in the fibers. By Lemma 3.1 (2), $\pi_2 M(DF) \neq 0$, and so $\pi_2 P(F) \neq 0$. □

Now let H be a finite subgroup of $SU(2)$. Set

$$S_0(H) = S^1 \times (SU(2)/H).$$

For certain H, namely C_n $(n > 2)$, $Q = D_4^*$ and $T = T_{12}^*$, there also exist non-trivial $(SU(2)/H)$-bundles over S^1 with structure group $N(H)/H$ (see §2). Denote these by $S_k(H)$ for certain $k > 0$, defined as the manifolds underlying the Mostert manifolds specified below:

$$S_1(C_n) = \begin{cases} S_{SU(2)}(C_n)_{x^2} & (\text{odd } n > 2) \\ S_{SO(3)}(C_{n/2})_{x^2} & (\text{even } n > 2) \end{cases}$$

$$S_1(Q) = S_{SO(3)}(D_4)_x$$

$$S_2(Q) = S_{SO(3)}(D_4)_{xy}$$

$$S_1(T) = S_{SO(3)}(T_{12})_y$$

(see §2).

Remark 3.4. Observe that $S_1(Q)$ may be defined using y or z in place of x, since x, y and z are conjugate in $N(D_4)$. Similarly yx may replace xy in defining $S_2(Q)$. Furthermore, $S_1(Q)$ and $S_2(Q)$ are the only two non-trivial $(SU(2)/Q)$-bundles over S^1 (up to diffeomorphism) [14]. Similarly $S_1(C_n)$ is the unique twisted bundle over S^1 with fiber the Lens space $L(n, 1)$ [15].

LEMMA 3.5. (1) *The manifolds $S_k(H)$ are classified up to diffeomorphism by k and H.*
(2) $\pi_2 S_k(H) = 0$.

Proof. The homotopy sequence of the fibration $SU(2)/H \to S_k(H) \to S^1$ yields a short exact sequence

$$1 \to H \subset \pi_1 S_k(H) \to \mathbf{Z} \to 1.$$

Since H is finite, it is the unique subgroup of $\pi_1 S_k(H)$ with quotient $\mathbf{Z}$. Thus H is an invariant of $S_k(H)$. To show that k is invariant one may compute $H_1 S_k(H)$ in the relevant cases. For example, $S_0(C_n) \neq S_1(C_n)$ since $H_1 S_0(C_n) = \mathbf{Z} \oplus \mathbf{Z}_n$ and $H_1 S_1(C_n) = \mathbf{Z} \oplus \mathbf{Z}_{gcd(2,n)}$ are not isomorphic ($n > 2$). Similarly the $S_k(Q)$ are distinguished by $H_1 S_k(Q)$ (which is $\mathbb{Z} \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$ for $k = 0$, $\mathbb{Z}$ for $k = 1$, and $\mathbb{Z} \oplus \mathbb{Z}_2$ for $k = 2$) and the $S_k(T)$ by $H_1 S_k(T)$ (which is $\mathbb{Z} \oplus \mathbb{Z}_3$ for $k = 0$ and $\mathbb{Z}$ for $k = 1$).

To prove (2) note that the short exact sequence above corresponds to the covering space $\mathbb{R} \times (SU(2)/H) \to S_k(H)$. Since the cover has trivial π_2, so does $S_k(H)$. □

Finally let

$$C(r, p) = r(S^1 \times S^3) \# p(S^1 \times P^3)$$

where kM denotes the connected sum of k copies of M.

LEMMA 3.6. (1) *$C(r, p)$ is classified up to diffeomorphism by r and p.*
(2) $\pi_2 C(r, p) = 0$.

Proof. We have $H_1 C(r, p) = \mathbb{Z}^{r+p} \oplus \mathbb{Z}_2^p$, proving (1). A transversality argument gives (2). □

The following topological classification is the principal result of this paper.

THEOREM 3.7. *Let M be a closed, connected oriented 4-manifold. Then M supports an effective action of a compact non-abelian Lie group G if and only if it is diffeomorphic to a manifold in one of the following classes:*

(1) S^4 *or* $\pm \mathbb{C}P^2$
(2) $C(r, p)$ $(r + p > 1)$
(3) $S_k(H)$
(4) $M(F)$ (*closed* F)
(5) $N(F)$ ($F = S^2$ *or* P^2)
(6) $P(F)$ (*bounded* $F \neq B^2$)

The manifolds in class (1) *are distinguished by their signatures σM, those in* (2) *by r and p, those in* (3) *by k and H, and those in* (4)–(6) *by F. The classes* (1)–(6) *are disjoint.*

Proof. First suppose that M supports an effective G-action. By Lemma 1.1 we may assume that $G = SU(2)$ or $SO(3)$. It follows that M is equivariantly diffeomorphic to a G-manifold in one of the three families $S_G(H)_n$, $I_G(H, K_0, K_1)$ or $F(f)$ listed in Theorem 2.4.

Case 1. $M = S_G(H)_n$. Then by definition (and Remark 3.4) M is diffeomorphic to $S_k(H^*)$, for appropriate k (where $H = H^*$ if $G = SU(2)$).

Case 2. $M = I_G(H, K_0, K_1)$, with H, K_0 and K_1 as tabled in Theorem 2.4. Then a case by case analysis shows that M is diffeomorphic to S^4, $\pm \mathbb{C}P^2$, $M(F)$ or $N(F)$ ($F = S^2$ or P^2). Indeed, it is evident from the structure of $I_G(H, K_0, K_1)$ as the union of two mapping cylinders that $I_{SU(2)}(1, SU(2), SU(2)) = S^4$ and $I_{SU(2)}(1, SO(2)^*, SU(2)) = \pm \mathbb{C}P^2$ (with orientation dependent upon the orientation of the orbit space). With a little more work one may show that $I_{SO(3)}(C_2, SO(2), O(2)') = \pm \mathbb{C}P^2$ and $I_{SO(3)}(D_4, O(2), O(2)') = S^4$: the former is the $SO(3)$-action induced from the natural $SU(3)$-action on $\mathbb{C}P^2$, and the latter is induced from the $SO(3)$-action by conjugation on $\mathbb{R}^5$ = symmetric traceless 3×3 matrices [1, pp. 42–44]. The remaining cases are handled using the following fibration result (essentially from [12]):

LEMMA 3.8. *If there is a closed subgroup K of G containing K_0 and K_1, then $I_G(H, K_0, K_1)$ fibers over G/K with fiber $I_K(H, K_0, K_1)$.*

Proof of the lemma. There is an isomorphism

$$G \times_K I_K(H, K_0, K_1) \to I_G(H, K_0, K_1)$$

defined by mapping $[g, [kH, t]]$ to $[gkH, t]$. Now apply Theorem II2.4 in [1]. □

Applying this lemma to the case $G = SU(2)$, $H = C_n$, $K_0 = K_1 = K = SO(2)^*$, we have $I_K(H, K_0, K_1) = S^2$, and so $I_{SU(2)}(C_n, SO(2)^*, SO(2)^*)$ is an S^2-bundle over $SU(2)/SO(2)^* = S^2$. Similarly $I_{SO(3)}(C_n, SO(2), SO(2)) = M(S^2)$ or $N(S^2)$ $(K = SO(2))$, and $I_{SO(3)}(D_{2n}, O(2), O(2)) = M(P^2)$ or $N(P^2)$ $(K = O(2))$.

Observe that the manifolds $N(S^2)$ and $N(P^2)$ do arise. For example $I_{SU(2)}(1, SO(2)^*, SO(2)^*) = N(S^2) = \mathbb{C}P^2 \# -\mathbb{C}P^2$, since it is by definition the union of two copies of the mapping cylinder of the Hopf map $S^3 \to S^2$. (Similarly $I_{SU(2)}(C_n, SO(2)^*, SO(2)^*) = N(S^2)$ and $I_{SO(3)}(C_n, SO(2), SO(2)) = M(S^2)$.) Also $I_{SO(3)}(D_2, O(2), O(2)) = N(P^2)$, since it is not a spin manifold (as seen by considering the self-intersection of the lift of the zero-section to the 2-fold cover $I_{SO(3)}(1, SO(2), SO(2))$, cf. §11 in [13] where it is shown that $I_{SO(3)}(D_{2n}, O(2), O(2)) = M(P^2)$ or $N(P^2)$ depending upon the parity of n).

Case 3. $M = F(f)$. If $f = 0$, then by definition $M = M(F)$ if $\partial F = \phi$ and $M = P(F)$ if $\partial F \neq \phi$. If $f > 0$, then a standard argument, cutting along the lifts of arcs in the orbit space joining pairs of fixed points (= 3-spheres in M) as shown in Fig. 3, shows that $M = C(r, p)$, where $p = rk(H_0 \partial F) - f$ and $r = rk(H_1 F) - p$ (cf. [8]). The cases when $r + p = 1$ correspond to $S^1 \times S^3 = S_0(1)$ $(p = 0)$ and $S^1 \times P^3 = S_0(C_2)$ $(p = 1)$.

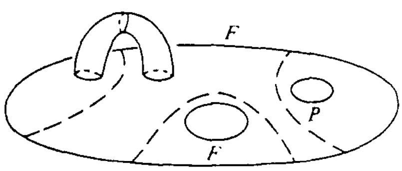


Fig. 3.

We have thus shown that M is diffeomorphic to one of the manifolds on the stated list. Conversely, if M is on the list, then it is evident from the actions discussed above that M supports an effective action of $SU(2)$ or $SO(3)$.

The classification of the manifolds within classes (2)–(6) was accomplished in Lemmas 3.1, 3.2, 3.5 and 3.6. It remains to show that the classes (1)–(6) do not overlap. This can be done by computing (in the appropriate cases) the signature σM, the second Stiefel–Whitney class $\omega_2 M$, and properties of the fundamental group $\pi_1 M$ (including the first Betti number $\beta_1 M$) and the second homotopy group $\pi_2 M$ (see lemmas). The results are given in Table 5.

This completes the proof of the theorem. □

In summary, Table 6 lists all effective $SU(2)$ and $SO(3)$-actions on the 4-manifolds classified above. (Note that $P(B^2) = M(P^2)$ (Remark 3.3) and $C(1,0) = S_0(1)$ and $C(0, 1) = S_0(C_2)$.) In this table, n can assume any integer value $\geqq 1$.

Table 5.

M	$\pi_2 M$	σM	$\beta_1 M$	$\pi_1 M$	$\omega_2 M$
CP^2	0	1			
$-CP^2$	0	-1			
S^4	0	0	0		
$C(r, p)$ $(r+p>1)$	0	0	>1		
$S_k(H)$	0	0	1		
$M(F)$ (closed F)	$\neq 0$			Finite or torsion free	0
$N(F)$ ($F = S^2$ or P^2)	$\neq 0$			Finite	$\neq 0$
$P(F)$ (bounded $F \neq B^2$)	$\neq 0$			Infinite with torsion	

Table 6.

$\pm CP^2$	$I_{SU(2)}(1, SO(2)^*, SU(2))$ $I_{SO(3)}(C_2, SO(2), O(2)')$
S^4	$I_{SU(2)}(1, SU(2), SU(2))$ $I_{SO(3)}(D_4, O(2), O(2)')$ $B^2(1)$
$C(r, p)$ $(r+p>0)$	$F(f)$ $(rk(H_1F) = r+p,\ rk(H_0\partial F) > p,\ f = rk(H_0\partial F) - p)$ Also $S_{SU(2)}(1)$ (if $r = 1$, $p = 0$) and $S_{SO(3)}(1)$ (if $r = 0$, $p = 1$)
$S_0(C_{2n+1})$	$S_{SU(2)}(C_{2n+1})$
$S_0(H)$ (H properly containing the center Z of $SU(2)$)	$S_{SO(3)}(H/Z)$
$S_1(C_{2n+1})$	$S_{SU(2)}(C_{2n+1})_{x^2}$
$S_1(C_{2n+2})$	$S_{SO(3)}(C_{n+1})_{x^2}$
$S_1(Q)$	$S_{SO(3)}(D_4)_{x \text{ or } y \text{ or } z}$
$S_2(Q)$	$S_{SO(3)}(D_4)_{xy \text{ or } yx}$
$S_1(T)$	$S_{SO(3)}(T_{12})_y$
$M(S^2)$	$I_{SO(3)}(C_n, SO(2), SO(2))$ $S^2(0)$
$N(S^2)$	$I_{SU(2)}(C_{2n-1}, SO(2)^*, SO(2)^*)$
$M(P^2)$	$I_{SO(3)}(D_{4n}, O(2), O(2))$ $B^2(0)$ $P^2(0)$
$N(P^2)$	$I_{SO(3)}(D_{4n-2}, O(2), O(2))$
$M(F)$ (closed $F \neq S^2, P^2$)	$F(0)$
$P(F)$ (bounded $F \neq B^2$)	$F(0)$

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