3-Dimensional Bordism

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In [10], C. Rourke gave an elementary proof of the following.

THEOREM 1 (Rohlin [9], Thom [11]). $\Omega_3^{SO}=0$ (*every closed oriented* 3-*manifold M is the oriented boundary of a compact oriented* 4-*manifold*).

Rourke's proof is by induction on Heegaard genus. He shows that if the genus of M is nonzero, then M is bordant to a disjoint union of manifolds of lower genus. The bordism is achieved by surgery on a carefully chosen link lying in the splitting surface of a Heegaard decomposition of M.

In this paper we show how to generalize Rourke's argument to give elementary proofs of the following two theorems.

THEOREM 2 (Rohlin [9], Thom [11]). $\Omega_3^{O}=0$ (*every closed* 3-*manifold is the boundary of a compact* 4-*manifold*).

THEOREM 3 (Milnor [6; 7]). $\Omega_3^{\mathrm{Spin}}=0$ (*every closed spin* 3-*manifold is the spin boundary of a compact spin* 4-*manifold*).

Compare Lickorish [5] for Theorem 2 and Kaplan [4] for Theorem 3.

The proof of Theorem 3 (or in particular Assertion 3, which is the chief contribution of this paper) gives an explicit construction of a family of simply-connected spin 4-manifolds with a given spin 3-manifold boundary. It is hoped that this family will be useful in the study of smooth closed simply-connected 4-manifolds and of invariants of 3-manifolds.

0. Preliminaries

We shall work in the smooth category. A *framing* $\mathbf{t}$ of a trivial vector bundle ϵ is a homotopy class of trivializations of ϵ, that is (up to homotopy) a list $t_1, \ldots, t_r$ of $r = \operatorname{rank}(\epsilon)$ linearly independent nonvanishing sections of ϵ. Write $\mathbf{t} = [t_1, \ldots, t_r]$.

Let M be an m-manifold. A framing of the restriction of the tangent bundle of M to a subset Z will be called a *tangential framing* of Z (in M). In particular, a *spin structure* on M is (for $m > 2$) a tangential framing of

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$M^{(2)}$, the 2-skeleton of some triangulation of M. See Milnor [7, p. 202]. M together with a spin structure $\mathbf{s}=[s_1,\ldots,s_m]$ is called a *spin manifold.* We say that M is the *spin boundary* of a spin manifold W (with spin structure $\mathbf{S}=[S_0,S_1,\ldots,S_m]$) if $M=\partial W$ and $\mathbf{S}\,|\,M=\mathbf{s}$ (i.e., $(S_1,\ldots,S_m)\,|\,M^{(2)}$ and $(s_1,\ldots,s_m)$ are homotopic). Spin manifolds M and N are *spin bordant* if the disjoint union $M\cup(-N)$ is a spin boundary. (Note that a spin manifold M has a natural orientation—i.e., tangential framing of $M^{(0)}$ which extends to $M^{(1)}$.)

Let Z be a submanifold of M with trivial normal bundle ν. A framing of ν will be called a *normal framing* of Z (in M). If in addition Z is embedded with trivial normal bundle in a submanifold F of M, and F has trivial normal bundle in M, then any framing of ν obtained by juxtaposing normal framings of Z in F and F in M will be called a *natural framing* of Z in M (relative to F).

Now let Z be a normally framed link of spheres in M. Denote by $M(Z)$ the manifold obtained from M by surgery along Z. If M lies in ∂W, then write W_Z for W with handles attached along Z. Observe that $\partial(W_Z)=(\partial W)(Z)$. In particular, M and $M(Z)$ are bordant, since $M\cup(-M(Z))=\partial(M\times I_{Z\times 1})$.

If M is spin and $\dim(Z)>1$, then $M(Z)$ can be given a spin structure for which M and $M(Z)$ are spin bordant. For example, if M is a connected sum $P\#Q$ with splitting sphere Z, then $M(Z)=P\cup Q$ inherits a spin structure by restriction (since $(P\cup Q)^{(2)}$ lies in $(P\#Q)^{(2)}$), and it is easily verified that $P\#Q$ and $P\cup Q$ are spin bordant.

For Z 1-dimensional, $M(Z)$ need not have a spin structure. There is, however, the following presumably well-known result. We state it for 3-manifolds (for use in our proof of Theorem 3) although it holds in higher dimensions as well. A proof is given in the appendix.

DEFINITION. A normally framed circle z in a spin manifold M is *spin* if its framing $\mathbf{n}$ is incompatible with the spin structure $\mathbf{s}$ on M (i.e., if $\mathbf{t}\neq\mathbf{s}\,|\,z$, where $\mathbf{t}$ is obtained from $\mathbf{n}$ by adding the oriented tangent to z).

LEMMA 0. *Let Z be a normally framed link of circles in a spin 3-manifold M. If each component of Z is spin, then $M(Z)$ has a spin structure for which M and $M(Z)$ are spin bordant.*

1. Rourke's Proof

Let F be a closed oriented surface and let x be a *curve* (i.e., a smoothly embedded circle) in F. Write $[x]$ for the class of x in $H_1(F;\mathbf{Z}_2)$. We say that x is *essential* if $[x]\neq 0$ (equivalently, if x is nonseparating). Two curves in F are *dual* if they intersect transversely in exactly one point. Note that curves which have duals are essential.

A union X of disjoint essential curves in F is called a *complete system of curves* in F if $F(X)$ is a sphere (for either normal framing of X). Two transverse complete systems are *dual* if they have a pair of dual components.

REMARK 1. *Every essential curve in a closed oriented surface is contained in a complete system of curves* (by the classification of surfaces).

A *Heegaard diagram* of genus g is a triple (F, X, Y), where X and Y are transverse complete systems of curves in a closed oriented surface F of genus g. Associated with the diagram is an oriented 3-manifold $F_{X,Y}$, obtained from $F \times I_{X \times 0 \cup Y \times 1}$ by capping off the boundary 2-spheres (X and Y are complete) with 3-balls. (Observe that F sits naturally in $F_{X,Y}$ as $F \times \{1/2\}$.) Every closed oriented 3-manifold M arises in this way, and the minimal genus of a diagram for M is called the *genus* of M. Note that if X and Y are dual, then the genus of $F_{X,Y}$ is less than the genus of F, by a standard handle cancellation argument.

Theorem 1 follows from the following.

LEMMA 1 (Rourke). *Let F be a closed oriented surface, and let X and Y be transverse complete systems of curves in F.*

(a) *For any complete system of curves Z in F (transverse to X and Y and naturally framed in $F_{X,Y}$),*

$$F_{X,Y}(Z) = F_{X,Z} \# F_{Z,Y}.$$

In particular, $F_{X,Y}$ and $F_{X,Z} \cup F_{Z,Y}$ are bordant.

(b) *There exists a sequence $Z_0, \ldots, Z_n$ of complete systems of curves in F, each dual to the next, with $Z_0 = X$ and $Z_n = Y$.*

Lemma 1(a) is Rourke's Lemma 1. The proof is easy: The surgery on $F_{X,Y}$ replaces each framed circle $z = S^1 \times B^2$ of Z by $B^2 \times S^1$. But $B^2 \times S^1$ can be viewed as the union of two 2-handles $B^2 \times S^1_{\pm}$, one attached to $F \times [0, 1/2]_{X \times 0}$ and the other to $F \times [1/2, 1]_{Y \times 1}$ along $z \times \{1/2\}$. This yields the desired connected sum with $F(Z)$ as the splitting 2-sphere. The last statement follows from observations in Section 0. Lemma 1(b) is immediate from the following assertion and Remark 1.

ASSERTION 1. *Let F be a closed oriented surface. If x and y are transverse essential curves in F, then there exists a sequence $z_0, \ldots, z_n$ of essential curves, each dual to the next, with $z_0 = x$ and $z_n = y$.*

This is in essence Rourke's Lemma 2. We prove a more general statement in Assertion 2 below.

Proof of Theorem 1. Let M be a closed oriented 3-manifold, and let (F, X, Y) be a Heegaard diagram for M of minimal genus g. If $g = 0$, then $M = S^3 = \partial B^4$, so assume $g > 0$. Choose a sequence $Z_0, \ldots, Z_n$ of complete systems as in Lemma 1(b). Applying Lemma 1(a) inductively, we see that $M = F_{X,Y}$ and $N = F_{Z_0, Z_1} \cup \cdots \cup F_{Z_{n-1}, Z_n}$ are bordant. But each F_{Z_{i-1}, Z_i} has genus less than g (since Z_{i-1} and Z_i are dual) and so, by induction, bounds a 4-manifold W_i. Thus $N = \partial(W_1 \cup \cdots \cup W_n)$ and so M bounds.

REMARK. An isotopy class of a complete system of curves is called a *cut system*. In [2], Hatcher and Thurston introduced a 2-complex X_g, whose vertices are the cut systems on a surface of genus g, and showed (using Cerf theory) that X_g is connected and simply connected. From this they derived a finite presentation for the mapping class group of the surface. Assertion 1 yields an elementary proof, by induction on g, of the connectedness of X_g. This, in turn, yields an even more direct proof of Theorem 1.

2. The Unoriented Case

To extend the result of Section 1 to the unoriented case, we must consider Heegaard diagrams (F, X, Y) of genus g for nonorientable 3-manifolds (see, e.g., Hempel [3]). F is taken to be a closed *nonorientable* surface of genus g (i.e., a connected sum of $g \geq 1$ Klein bottles), and X and Y are transverse *complete systems of curves* on F, defined as in Section 1 with the appropriate modified definition of essential curves: An *essential curve* in F is a nonseparating 2-sided curve x with $[x] \neq \omega$ when $g > 1$, where $\omega \in H_1(F; \mathbf{Z}_2)$ is the dual of the first Stiefel–Whitney class of F. (The condition $[x] \neq \omega$ is equivalent to the nonorientability of $F - x$.) By the classification of surfaces, once again, one has the following.

REMARK 2. *Remark 1 holds for arbitrary closed surfaces.*

The definition of $F_{X,Y}$ is exactly as in Section 1. One readily verifies that every closed nonorientable 3-manifold is diffeomorphic to some $F_{X,Y}$. With these definitions, Lemma 1 goes through in the nonorientable case.

LEMMA 2. *Lemma 1 holds for arbitrary closed surfaces.*

The proof of 1(a) is unchanged, and 1(b) is a consequence of the following Assertion and Remark 2. (It is here that the condition $[x] \neq \omega$ for curves in X and Y is used.)

ASSERTION 2. *Assertion 1 holds for arbitrary closed surfaces.* [This result is used in a forthcoming paper by D. Gabai and the second author on the classification of maps of nonorientable surfaces.]

Proof. Let $x \cdot y$ denote the number of points of $x \cap y$. If $x \cdot y = 0$, then write $x \# y$ for any *band sum* of x and y along an arc disjoint from $x \cup y$. In general, write $x + y$ for the *double point sum* of x and y (with respect to any choice of orientation), obtained from $x \cup y$ by smoothing the double points (see Figure 1). Note that $[x \# y] = [x + y] = [x] + [y]$. Thus if $x \# y$ or

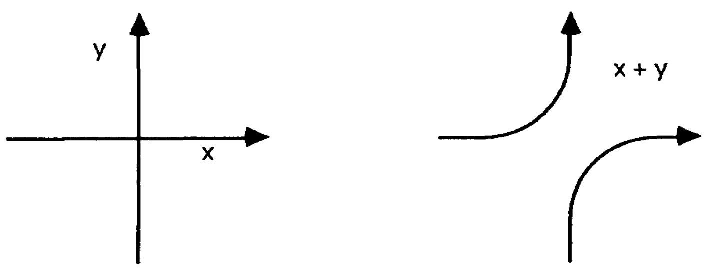

Figure 1

$x+y$ is an essential curve and x and y are 2-sided, then x or y is essential as well.

If $x \cdot y = 0$, then we show how to find an essential curve z dual to both x and y, provided $g > 1$. (If $g = 1$ then $x = y$.) It is easy to find some curve z dual to both, as neither separates F. (In particular, $[z] \neq 0$.) If z is 1-sided, then a regular neighborhood of $x \cup y \cup z$ is a twice-punctured Klein bottle (Figure 2). It follows that $F-(x \cup y \cup z)$ is nonorientable, since both $F-x$

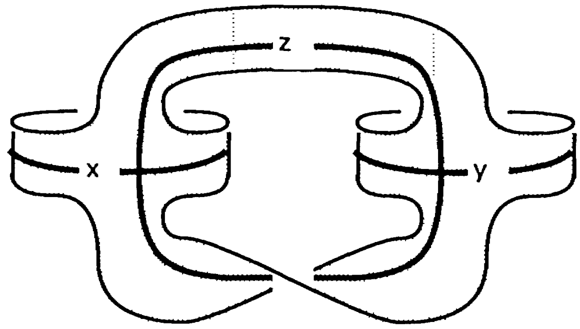

Figure 2

and $F-y$ are nonorientable. Thus z may be modified (by band summing with a 1-sided curve in $F-(x \cup y \cup z)$) to become 2-sided. Finally, arrange that $[z] \neq \omega$ by using $x+z$ in place of z if necessary.

If $x \cdot y = 1$, then x and y are dual and there is nothing to show.

Now assume that $x \cdot y > 1$. A curve z (transverse to x and y) with $x \cdot z$ and $y \cdot z$ both less than $x \cdot y$ will be called *admissible.* It suffices by induction to find an admissible essential curve.

A component of $y-x$ will be called a *singular arc* of y if its endpoints are oppositely oriented, in the following sense: A choice of orientations on x and y yields orientations on the endpoints of the arc, and these can be compared within a regular neighborhood of x, which is orientable since x is 2-sided. Similarly define the singular arcs of x.

Now let y_1 be a component of $y-x$ and N be a regular neighborhood of $x \cup y_1$. There are evidently four possibilities for N, shown in Figure 3(a)–(d). These can be classified by the orientability of N (distinguishing (a) and (b) from (c) and (d)) and the singularity of y_1 (distinguishing (a) and (c) from (b) and (d)).

For (a) and (b), consider the admissible 2-sided curves z and z' shown, with $x = z \# z'$ in (a) and $z + z'$ in (b). Since x is essential, either z or z' is as well. For (c) we have $[z] = [x]$, and so z is 2-sided and essential (and evidently admissible). Finally, case (d) reduces to (a) or (c) by swapping roles of x and y, as it is evident that some arc of x is singular. The proof of the assertion is complete. □

Theorem 2 now follows as in Section 1, except that (in the nonorientable case) the induction begins with genus 1 when $M = S^1 \tilde{\times} S^2 = \partial(S^1 \tilde{\times} B^3)$ (cf. Hempel [3]).

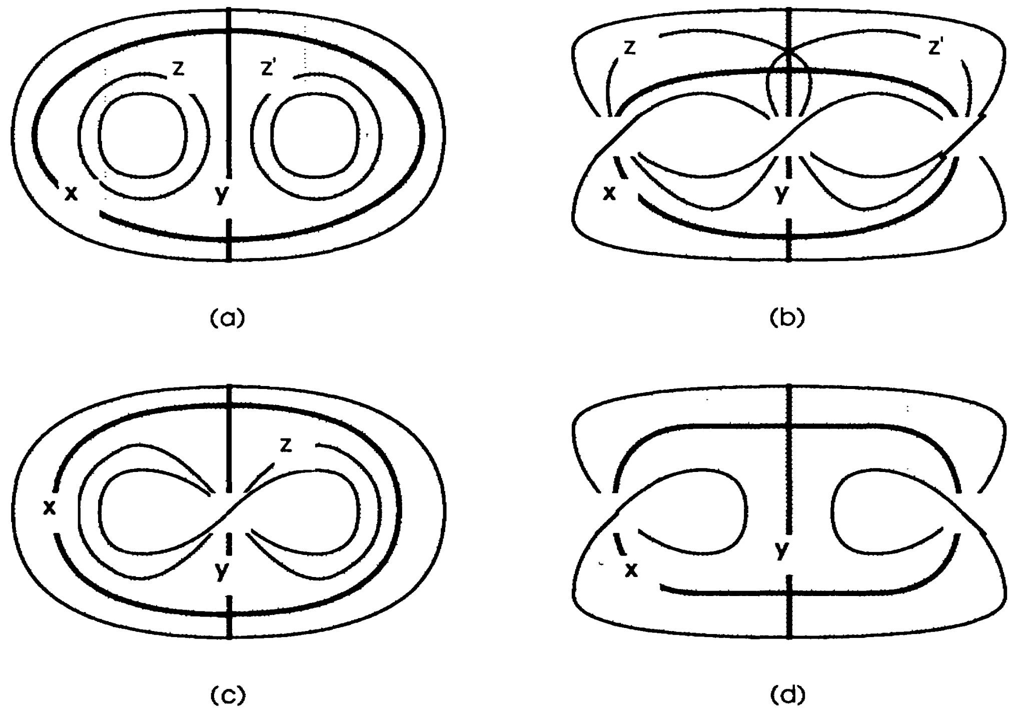

Figure 3

3. The Spin Case

Let F be a closed oriented surface in a spin 3-manifold M, and let x be a curve in F naturally framed in M (relative to F). It turns out that x is spin if and only if $[x]$ lies in the kernel of a certain *quadratic form* on $H_1(F;\mathbf{Z}_2)$. (See, e.g., [1, pp. 54–56] for a general discussion of $\mathbf{Z}_2$-valued quadratic forms.)

To be precise, set $f(x)=1$ if x is spin, and $f(x)=0$ otherwise (for each curve x in F). Extend f to a $\mathbf{Z}_2$-valued function on the set of all embedded 1-manifolds in F by summing over components. Now define

$$q(x)=f(x)+|x| \pmod 2,$$

where $|x|$ denotes the number of components of x. Thus a curve x is spin if and only if $q(x)=0$, by definition.

The following result is essentially due to Pontryagin [8, pp. 103–106].

PROPOSITION. *q defines a quadratic form on $H_1(F;\mathbf{Z}_2)$. That is, $q(x)=q(y)$ if $[x]=[y]$, and (for x and y transverse)*

(*) $$q(x+y)=q(x)+q(y)+x\cdot y \pmod 2.$$

In particular (for x and y disjoint), $q(x\# y)=q(x)+q(y)$.

Note: The + on the left side of the equality in (*) denotes double point sum, and • denotes geometric intersection number (see the proof of Assertion 2).

Proof. Evidently,

(1) $$f(x+y)=f(x)+f(y).$$

Furthermore, it is not hard to show that

(2) $$f(x)=|x| \pmod 2 \quad \text{if } [x]=0$$

and

(3) $$|x+y|=|x|+|y|+x\cdot y \pmod 2.$$

For (2), let S be a surface in F with $\partial S=x$. Note that the mod 2 Euler characteristic $\chi(S)$ of S is the obstruction to extending the natural stable framing of ∂S to a stable framing of S. Thus $f(x)=\chi(S)$. But $\chi(S)=|\partial S|$ (mod 2), and (2) follows.

For (3), embed $x\cup y$ in $F\times I$ with x in $F\times 0$ and y in $F\times 1$. Now alter this embedding to an embedding of $x+y$ in $F\times I$ by a sequence of $x\cdot y$ ambient surgeries, one for each double point of x and y in F. Each surgery changes the number of components by one, and so (3) follows.

It is now evident that q is well defined on $H_1(F;\mathbf{Z}_2)$. For if $[x]=[y]$, then $[x+y]=0$ and $x\cdot y=0$, and so

$$\begin{aligned} f(x)+f(y) &= f(x+y) && \text{by (1)} \\ &= |x+y| && \text{by (2)} \\ &= |x|+|y|+x\cdot y && \text{by (3)} \\ &= |x|+|y|. \end{aligned}$$

Thus $q(x)=f(x)+|x|=f(y)+|y|=q(y)$ (mod 2).

A similar argument using (1) and (3) establishes the quadratic identity (*). The last statement follows since $[x\#y]=[x+y]$. □

We call q the form *induced* on F by the spin structure on M. Observe that if $M=F_{X,Y}$ (as in §1) then q vanishes on the components of X and Y, since each bounds a disc away from F (cf. the proof of (2) in the Proposition). It follows that q has Arf invariant 0 [1, p. 56]. This motivates the following definition.

DEFINITION. A *quadratic surface* (F,q) is a closed oriented surface F together with a quadratic form q on $H_1(F;\mathbf{Z}_2)$ (as in the Proposition) of Arf invariant 0. A curve x in (F,q) is *spin* if $q(x)=0$, and *essential* if it is spin and nonseparating. A *complete system of curves* X in (F,q) is a union of disjoint essential curves in (F,q) with $F(X)=S^2$.

One then has the following analogue of Remark 1 for quadratic surfaces.

REMARK 3. *Any essential curve x in a quadratic surface (F,q) lies in a complete system of curves in (F,q).*

Proof. By Remark 1, x lies in a complete system $x_1,\dots,x_g$ in F (forgetting q). Suppose that $q(x_i)=1$ for some i. Choose another complete system $y_1,\dots,y_g$ in F such that $x_i\cdot y_j=\delta_{ij}$ (i.e., the x's and y's form a symplectic basis). If $q(y_i)=0$ then replace x_i by y_i. If $q(y_i)=1$ then, since the Arf invariant of q is 0, there is some $j\neq i$ with $q(x_j)=q(y_j)=1$ ([1, p. 54]). In this case, replace x_i by $x_i\#x_j$ and x_j by $y_i\#y_j$ (embedded disjointly). Continue until all the curves in the system are spin. □

Theorem 3 now follows, exactly as in Section 1, from the following.

LEMMA 3. *Lemma 1 holds for quadratic surfaces* (F, q), *with the following addendum to 1(a): For any spin structure on* $F_{X,Y}$ *which induces* q, *there are spin structures on* $F_{X,Z}$ *and* $F_{Z,Y}$ *such that* $F_{X,Y}$ *and* $F_{X,Z} \cup F_{Z,Y}$ *are spin bordant.*

The proof of 1(a) is unchanged. For the addendum, note that Lemma 0 provides a spin bordism between $F_{X,Y}$ and $F_{X,Y}(Z) = F_{X,Z} \# F_{Z,Y}$ (with an appropriate spin structure). But, as observed in Section 0, the latter is spin bordant to $F_{X,Z} \cup F_{Z,Y}$ (with the inherited spin structure). Finally, 1(b) is a consequence of the following assertion and Remark 3.

ASSERTION 3. *Assertion 1 holds for quadratic surfaces* (F, q).

Proof. (Note that the assumption that q has Arf invariant 0 is not used in the proof.) We adopt the notation and terminology of the proof of Assertion 2. Observe that if z and z' are dual and $z + z'$ is essential, then z or z' is essential.

If $x \cdot y = 0$, then, as in the proof of Assertion 2, there is a curve z dual to x and y. But then $z' = z + x$ is also dual to x and y, and z and z' are dual to each other. Since $x = z + z'$ (appropriately oriented) is essential, one of z or z' is essential.

If $x \cdot y = 1$, then there is nothing to show. If $x \cdot y = 2$, then one can easily find a curve z dual to x and y by considering the two possibilities for a regular neighborhood of $x \cup y$ and using the fact that neither x nor y separates. This case now proceeds as in the case when $x \cdot y = 0$.

Finally, assume that $x \cdot y > 2$. It suffices by induction to find an admissible essential curve. Recall that an arc (component) of $x \Delta y = (x \cup y) - (x \cap y)$ is *singular* if its endpoints are oppositely oriented with respect to some orientation on x and y. There are two cases.

Case 1. Some arc in $x \Delta y$ (say $y_1 \subset y$) is nonsingular.

Then a regular neighborhood of $x \cup y_1$ looks like Figure 3(b). The two admissible curves z and z' shown are dual and $x = z + z'$. Thus one is essential.

Case 2. All arcs in $x \Delta y$ are singular.

In particular, a neighborhood of $x \cup y_1$, for any arc y_1 in $y - x$, is a pair of pants with waist parallel to x [see Figure 3(a)]. The other two components of ∂P are called *cuffs of* x [z and z' in Figure 3(a)], and the two cuffs of P are said to be *twins*. Similarly define cuffs of y. All cuffs are evidently admissible.

Suppose that some cuff z (say, of x) is spin. Then its twin z' is spin as well. Indeed, $x = z \# z'$ and so $0 = q(x) = q(z) + q(z')$. But one of the twins is nonseparating, since x is nonseparating, and so one is essential. Thus we may assume that every cuff is nonspin.

Let y_1 and y_2 be adjacent arcs of $y - x$, and let N be a regular neighborhood of $x \cup y_1 \cup y_2$. Since y_1 and y_2 are singular, N is a disc with three holes as shown in Figure 4. Observe that all the boundary components z_1, z_2, z_3, and z_4 of N are cuffs. Now both of the admissible curves $u = z_1 \# z_3$ and $v = z_2 \# z_3$

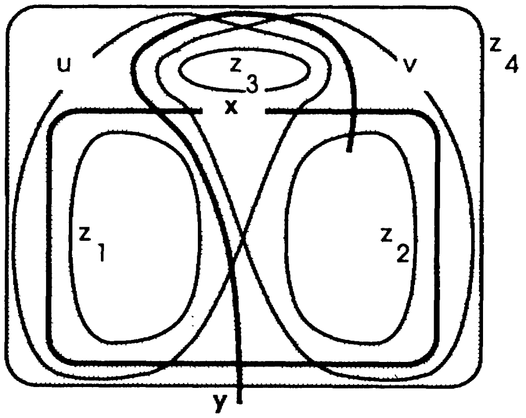

Figure 4

shown in Figure 4 are spin, since the z_i are nonspin. But at least one of u or v is nonseparating because $[u]+[v]=[x]\neq 0$, and so one is essential. □

REMARK. As in the remark at the end of Section 1, Assertion 2 shows that a certain *spin subcomplex* of the Hatcher–Thurston complex X_g is connected, and this yields an even more direct proof of Theorem 3. A similar remark can be made in the unoriented case.

Appendix. Proof of Lemma 0

Choose a component z of Z, and set $W=(M\times I)_{z\times 1}$; W is obtained by attaching a 2-handle $H=B^2\times B^2$ to $M\times I$, identifying $H_-=S^1\times B^2$ with $N\times 1$ (where N is a tubular neighborhood of z, identified with $S^1\times B^2$ by the normal framing **n**). Set $H_+=B^2\times S^1$. Note that (after smoothing corners) $F=H_+\cap H_-$ is a smooth torus which is the common boundary of H_+ and H_- (see Figure 5).

Extend the spin structure s on M to a framing of $M=M\times 0$, and then (by adding the outward normal to $M\times 0$ and extending trivially) to a tangential framing **T** of $M\times I$ in W. Similarly, extend the framing **n** on z (after adding the oriented tangent to z and the outward normal to $M\times 0$) to a tangential framing **T**′ of $N\times I$ in W.

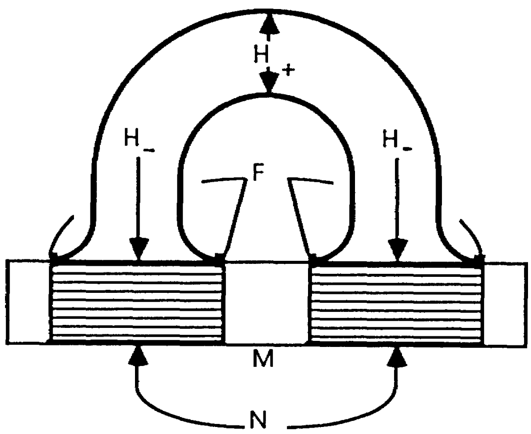

Figure 5

Now, dropping the normals, **T** and **T**′ induce distinct tangential framings **t** and **t**′ on F in $H_{\pm}$ (since **n** and s are incompatible) which both extend to H_-. But there are only two such framings on F, and one also extends to H_+ (since $\pi_1\,\mathrm{SO}(3)=\mathbf{Z}_2$). Evidently **t**′ does not extend to H_+. Indeed, using polar coordinates (r,θ,φ) on $H_+=B^2\times S^1$, $\mathbf{t}'\,|\,(S^2\times 0)=[\partial/\partial\theta,\partial/\partial r,\partial/\partial\varphi]$ does not even extend to $B^2\times 0$. Thus **t** extends to H_+, giving an extension of **T** to $(M\times I)\cup H_+\simeq W-$point, and thus (by restriction) a spin structure **S** on W with $\mathbf{S}\,|\,M=\mathrm{s}$. Hence M and $M(z)$ are spin bordant.

The lemma follows by induction on the number of components of Z. □

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