

A TOPOLOGICAL MENAGERIE

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Some of the party games that young teenagers play have surprisingly rich mathematical content. “Entanglement” is one such game in which couples are linked with strings tied tightly to their wrists, as shown in Figure 1a,

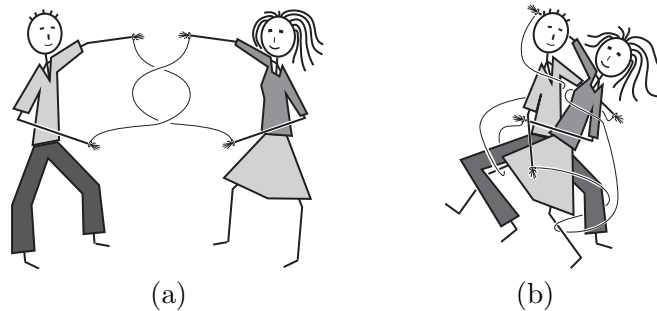


Figure 1: Entanglement.

and then challenged to disentangle themselves, often leading to Figure 1b. In fact there is a quick solution (needless to say missing the point of the game): simply push a bit of the girl’s string under the boy’s at his wrist, pass the resulting loop over his hand, and then pull it free on the other side, as shown in Figure 2.



Figure 2: Disentanglement.

One way to formulate this mathematically is to ask for an isotopy to remove the meridian loop m from a rigid wire w embedded in the 3-sphere, as shown in Figure 3a. (The wire represents the boy with his attached string, where the upper loop of the wire is his right hand, while the meridian represents the girl with her string.) Of course this is easy: just slide m along w to a point just below the upper loop of w , and then stretch it out and pull it off the top. Or put differently, observe that w is isotopic to the trivial wire $\circ\text{---}\circ$ and so the meridian slips right off.

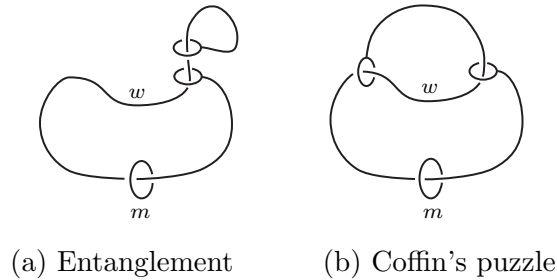


Figure 3

A similar puzzle attributed to Steward Coffin [2] was considered by Inta Bertuccioni in a recent issue of the Monthly [1]. Here the wire w is configured slightly differently (as in Figure 3b) and there is in fact no solution, that is, m cannot be isotoped off w . Bertuccioni proves this by an explicit calculation showing that m represents a nontrivial element in the fundamental group of the complement of w .

The purpose of this note is to give a more conceptual proof of the impossibility of solving Coffin's puzzle. We then generalize the proof, using an elementary but nontrivial result in knot theory to show that all but one of the vast "menagerie" of possible puzzles suggested by entanglement and Coffin's puzzle are also unsolvable.

First consider Coffin's puzzle. The wire w consists of two unknots λ joined by an arc α , as illustrated in Figure 4a. From this one obtains a knot k by banding the components of λ together along α , shown in Figure 4b.

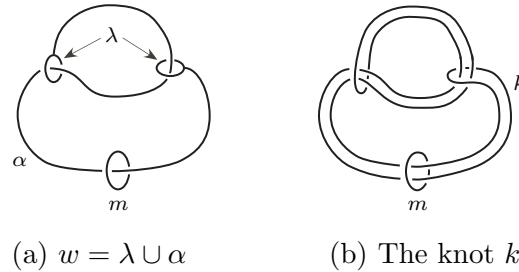


Figure 4

Actually there are infinitely many knots that can be formed in this way, for there may be twists in the band. The one pictured is readily seen to be the square knot.

Now observe that if m could be isotoped off w , then it certainly could be isotoped off k , since the complement of k contains the complement of w (with α thickened a bit). But then the lift \tilde{k} of k to the infinite cyclic cover of the complement of m would consist of an infinite number of copies of k . (Note that the complement of m is a solid torus, and the cover is an infinite solid cylinder.) However, it is easily seen that the components of \tilde{k} are *unknotted*, contradicting the fact that k is a nontrivial knot (the square knot). Indeed, viewing k as the closure of a tangle with axis m , the link \tilde{k}

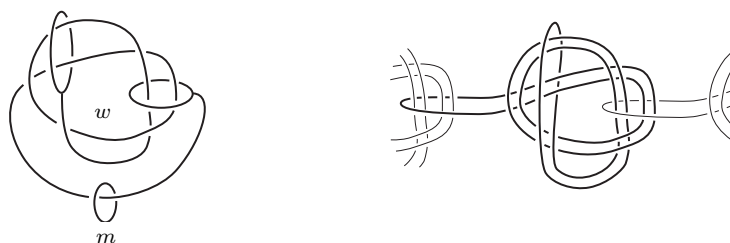
is obtained by composing infinitely many copies of this tangle, as illustrated in Figure 5 (see for example [4]).



Figure 5: \tilde{k} in the infinite cyclic cover.

Thus each component of \tilde{k} looks like a long worm that can be shrunk by pushing from its “free” end.

For a general puzzle in the menagerie we allow any embedding of the wire for which λ is an unlink, while the arc α can be arbitrary. One such puzzle is shown in Figure 6a.



(a) A beast in the menagerie

(b) The associated lift \tilde{k}

Figure 6

The foregoing argument demonstrates that there is no hope for a solution unless the associated knot k is trivial, because each component of the lift \tilde{k} is unknotted (as seen in Figure 6b; in general slide m close to an endpoint of α before taking the cover). However, by a theorem of Marty Scharlemann [5], k is trivial if and only if the arc α is isotopic (fixing λ) to a trivial arc. Thus “most” of the puzzles in the menagerie, indeed all but those equivalent to entanglement, have no solution.

From a practical point of view one might ask how to recognize when α is trivial. This is actually a difficult question. However there is a simple test for the triviality of the *homotopy* class of α : The fundamental group of the complement of λ is free of rank two, with generators x and y corresponding to the two components of λ . The arc α , oriented from the x -loop to the y -loop of λ , determines a word in x and y , also denoted α . Since the arc α can be made to spiral around λ at its endpoints, its homotopy class corresponds to an *equivalence class* of words: $\alpha \sim \beta$ if and only if there exist integers m and n for which $\alpha = x^m \beta y^n$. Thus *the arc α is homotopically trivial if and only if the word $\alpha \sim 1$* . For example, the entanglement puzzle has $\alpha = xy \sim 1$ as expected, whereas Coffin’s puzzle has $\alpha = yx \not\sim 1$, and the puzzle in Figure 6a has $\alpha = yxyx \not\sim 1$.

Of course there are many homotopically trivial arcs α that are isotopically nontrivial, or equivalently, whose associated knots are nontrivial. Consider, for example, the puzzle in Figure 7, which has $\alpha = xy \sim 1$.

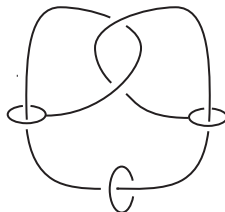


Figure 7: A homotopically trivial beast.

The associated knot has Jones polynomial $t^{-5} - t^{-4} - t^{-1} + 2 - t + t^2 + t^5 - t^6$ (see [3]; its Alexander polynomial is trivial), so this puzzle has no solution.

REFERENCES

- [1] I. Bertuccioni, A topological puzzle, this MONTHLY **110** (2003) 937–939.
- [2] P. Van Delft and J. Botermans, *Creative Puzzles of the World*, Abrams, New York, 1975.
- [3] V. F. R. Jones, A polynomial invariant for knots via von Neumann algebras, *Bull. Amer. Math. Soc.* **12** (1985) 103–111.
- [4] D. Rolfsen, *Knots and Links*, Publish or Perish Inc., Berkeley, CA, 1976.
- [5] M. Scharlemann, Smooth spheres in \mathbb{R}^4 with four critical points are standard, *Inventiones Math.* **79** (1985) 125–141.

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