ALGEBRAIC TOPOLOGY: HOMOLOGY AND COHOMOLOGY THEORY

Math 525
Bryn Mawr College
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0. Introduction

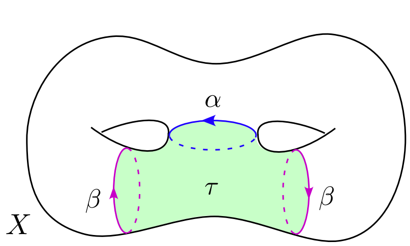
Homology theory assigns to any topological space X a sequence $H_0(X), H_1(X), H_2(X), \ldots$ of abelian groups called the homology groups of X. In rough geometric terms, these groups record the "holes" in X. In particular, the elements in $H_n(X)$ are the n-dimensional holes, each of which can be represented (in many ways) by n-cycles, oriented n-dimensional objects *without boundary* mapped into X to surround the hole.[†] The sum of two holes is represented by the formal sum of their representative cycles, so the identity element $0 \in H_n(X)$ is represented by the empty n-cycle.

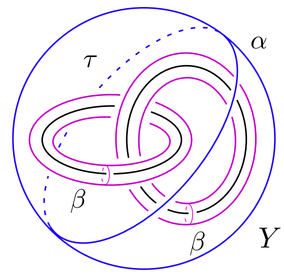
To make this more precise, we introduce an equivalence relation $\sim$ on cycles: First extend the notion of n-cycles to n-chains, oriented n-dimensional objects that *may have boundary*. The term *chain* is used because such objects in practice are built up as chains of simpler objects, e.g. triangles, tetrahedra, etc. More on this in §2. Now two n-cycles α and β are homologous, written $\alpha \sim \beta$, if their difference $\alpha - \beta = \alpha + (-\beta)$ (where $-\beta = \beta$ with its orientation reversed) is the boundary $\partial\tau$ of some $(n+1)$-chain τ.

We write $\overline{\alpha}$ for the equivalence class of a cycle α and call it the homology class of α; this is one of the "holes" above. Thus by definition $\overline{\alpha} = \overline{\beta} \iff \alpha \sim \beta \iff \alpha - \beta = \partial\tau$ for some $(n+1)$-chain τ. In particular $\overline{\alpha} = 0 \iff \alpha$ bounds an $(n+1)$-chain, and in this case we say that α is null-homologous.

This is illustrated in the figure below. On the left X is a 2-holed torus, α is the blue loop, β is the pair of purple loops, and τ is the green shaded surface that α and β cobound, showing that $\overline{\alpha} = \overline{\beta}$ in $H_1(X)$. On the right Y is the complement in $\mathbb{R}^3$ of the 2-component Hopf link (drawn in black), α is the blue sphere, β is the pair of purple tori, and τ is the 3-dimensional region between α and β, showing that $\overline{\alpha} = \overline{\beta}$ in $H_2(Y)$. It is left to the reader to sort out the orientations (given by arrows on the left, and implicitly by outward pointing normals on the right).

[†] These objects need not be spherical, as for the homotopy groups $\pi_n(X)$, nor even connected. But they should be compact. For example, the 1-cycles and the 2-cycles, respectively, can be viewed as finite collections of oriented loops, and of closed oriented surfaces, mapped into X.



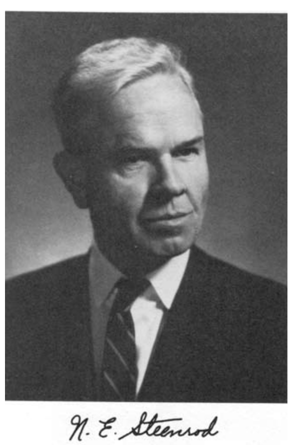


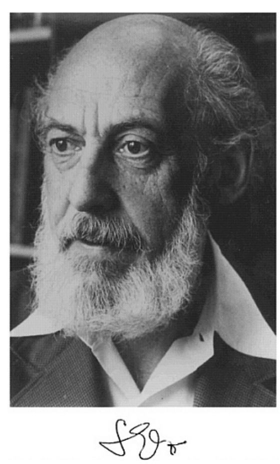
Below is a collage of just a few of the leading figures in the development of algebraic topology during the first half of the twentieth century:

Henri Poincaré

Emmy Noether

Heinz Hopf



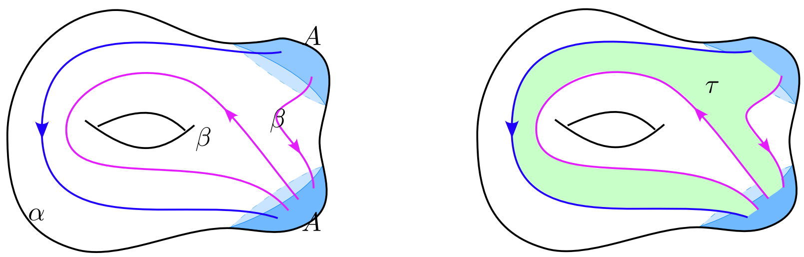


Many different homology theories were introduced in the first half of the 20th century, but it was not until the mid 1940s through the joint work of Samuel Eilenberg and Norman Steenrod (the last pair pictured above) that topologists understood how to relate these theories, and for that matter, how to define what a homology theory really is.

1. Formal Approach

This section presents the axiomatic approach to homology theory due to Eilenberg and Steenrod† as well as some fundamental tools for calculating homology groups from the axioms. As it turns out, it is convenient to assign homology groups to all pairs (X, A) of topological spaces with $A \subset X$ (where a single space X is identified with the pair $(X, \varnothing)$) or at least an appropriately restricted class of admissible pairs such as simplicial or cellular pairs (see §2, or [ES, I§1] for more details).

From the geometric perspective, the elements of $H_n(X, A)$ can be viewed as equivalence classes $\overline{\alpha}$ of oriented n-chains α in X with boundary $\partial\alpha$ in A, where $\overline{\alpha} = \overline{\beta}$ if and only if there is an oriented $(n+1)$-chain τ in X with boundary $\partial\tau = \alpha - \beta + \gamma$ for some chain γ in A, also written $\partial\tau \equiv_A \alpha - \beta$. In particular α and $\alpha + \gamma$ for any n-chain γ in A represent the same relative homology class. This is illustrated below when X is a torus T^2 and A is a pair of disks in X.



To state the axioms, we need the notion of a map of pairs $f : (X, A) \to (Y, B)$, meaning a map $f : X \to Y$ for which $f(A) \subset B$ (in this context "maps" are always assumed continuous, while maps between groups are homomorphisms). Two such maps are homotopic if they are homotopic through maps of pairs, i.e. each stage of the homotopy carries A into B. Write $\mathrm{Map}((X, A), (Y, B))$ for the set of all such maps, and $[(X, A), (Y, B)]$ for the set of homotopy classes of such maps. For example $\pi_1(X, x_0) = [(I, \partial I), (X, x_0)]$ (where $I := [0, 1]$). We may also encounter triples (X, A, B) of spaces, where $B \subset A \subset X$, and triads $(X; A, B)$, where $X = A \cup B$ but A and B need not be nested. Of particular importance are excisive triads $(X; A, B)$, requiring X to be covered by the *interiors* of A and B. Maps of triples or triads are defined as for pairs.

In addition, we need the fundamental notions of exact sequences of abelian groups, and more generally for later use, chain complexes and chain maps between chain complexes. Note: Analogous definitions can be made in any category of modules, but we will not need these for our purposes.

Definition A sequence $\cdots \to C_{n+1} \xrightarrow{\partial_{n+1}} C_n \xrightarrow{\partial_n} C_{n-1} \to \cdots$ of abelian groups joined by homomorphisms is exact at C_n if $\mathrm{im}(\partial_{n+1}) = \ker(\partial_n)$, and is exact if it is exact at each group in the sequence. More generally, this sequence is called a chain complex if $\mathrm{im}(\partial_{n+1}) \subset \ker(\partial_n)$, or equivalently $\partial_n \partial_{n+1} = 0$, for each n (so an exact sequence is a special kind of chain complex).

A chain map from this chain complex to another $\cdots \to C'_{n+1} \xrightarrow{\partial'_{n+1}} C'_n \xrightarrow{\partial'_n} C'_{n-1} \to \cdots$ is a sequence of maps $f_n : C_n \to C'_n$

$$\begin{array}{ccccccccc}
\cdots & \longrightarrow & C_{n+1} & \xrightarrow{\partial_{n+1}} & C_n & \xrightarrow{\partial_n} & C_{n-1} & \longrightarrow & \cdots \\
 & & \downarrow{\scriptstyle f_{n+1}} & & \downarrow{\scriptstyle f_n} & & \downarrow{\scriptstyle f_{n-1}} & & \\
\cdots & \longrightarrow & C'_{n+1} & \xrightarrow{\partial'_{n+1}} & C'_n & \xrightarrow{\partial'_n} & C'_{n-1} & \longrightarrow & \cdots
\end{array}$$

that make the diagram commute: $\partial'_n f_n = f_{n-1} \partial_n$ for all n.

† announced in Proc. N.A.S. **31** (1945) 117–120, and spelled out in their 1952 book [ES]: *Foundations of Algebraic Topology*, Princeton University Press

1.1 The Eilenberg-Steenrod Axioms (1945)

A homology theory consists of two functions H and ∂:

- H assigns to each (admissible) pair (X, A) a sequence of abelian groups $H_n(X, A)$ for all $n \in \mathbb{Z}$ (equal to zero for all $n < 0$, see [ES, I§3]) called the homology groups of (X, A), and to each map of pairs $f : (X, A) \longrightarrow (Y, B)$ a sequence of group homomorphisms $f_n : H_n(X, A) \longrightarrow H_n(Y, B)$.

Remark The elements in $H_n(X, A)$ are called relative homology classes, while those in $H_n(X)$ are absolute classes. The maps f_n, usually denoted f_* with implicit dependence on n, are called the maps induced by f; if they are all isomorphisms, then f is called a homology equivalence.

- ∂ assigns to each pair (X, A) a sequence of group homomorphisms $\partial_n : H_{n+1}(X, A) \longrightarrow H_n(A)$ (usually just written ∂ with implicit dependence on n) called the boundary maps.

These must satisfy the following seven axioms:

①–② (functoriality) $\mathbb{1}_* = \mathbb{1}$ (where $\mathbb{1}$s denote identity maps) and $(fg)_* = f_* g_*$
③ (naturality) $\partial f_* = (f|A)_* \partial$
④ (homotopy) $f \simeq g \implies f_* = g_*$
⑤ (exactness) For any pair (X, A), the sequence

$$\cdots \longrightarrow H_{n+1}(X, A) \xrightarrow{\partial} H_n(A) \longrightarrow H_n(X) \longrightarrow H_n(X, A) \xrightarrow{\partial} H_{n-1}(A) \longrightarrow \cdots$$

is exact, where unlabeled maps are induced by inclusions.

⑥ (excision) If $\overline{U} \subset A^\circ$, then the inclusion $(X - U, A - U) \hookrightarrow (X, A)$ is a homology equivalence.†
⑦ (dimension) $H_n(\text{pt}) = 0$ for all $n \neq 0$ (where pt denotes a space consisting of a single point)

Remark The group $G = H_0(\text{pt})$ is called the coefficient group of the theory. Typically one constructs theories with $G = \mathbb{Z}$, and then there is a formal procedure for producing associated theories with any other coefficient group. If $G \neq \mathbb{Z}$, we generally write $H_n(X, A; G)$ for $H_n(X, A)$.

There are many homology theories. The best known are

- simplicial homology (defined for simplicial pairs; good for understanding basic notions)
- cellular homology (defined for CW-pairs; great for calculations)
- singular homology (defined for all pairs; good for verifying the axioms)
- Čech homology (defined for all pairs; good for weird spaces)

These theories (and any others that satisfy the ES axioms) are equivalent on the category of CW-pairs that are admissible for both theories, meaning they assign the same groups and homomorphisms once the coefficient group is fixed (but may differ on more general spaces).

Remarks ① Continuing the geometric perspective above, one can define $f_*(\overline{\alpha}) := \overline{f(\alpha)}$ and $\partial\overline{\alpha} := \overline{\partial\alpha}$. It is instructive to verify the exactness and excision axioms from this point of view. We leave this as an exercise for the reader.

② Under suitable conditions on (X, A), explained more fully in §3, the 'relative' homology groups $H_n(X, A)$ are isomorphic to (a reduced version of) the absolute groups $H_n(X/A)$, where X/A is the quotient space obtained from X by collapsing A to a point. But in general, $H_n(X, A)$ and $H_n(X/A)$ need not be related.

† Setting $B = X - U$, this can be reformulated as a statement about triads: If $(X; A, B)$ is excisive, then the inclusion $(B, A \cap B) \hookrightarrow (X, A)$ is a homology equivalence.

③ Excluding the dimension axiom and allowing $n < 0$, we get the so-called generalized homology theories. These have played a major role in many areas of research in topology, and include bordism theory, K-theory and stable homotopy theory.

④ Cohomology theory is defined by reversing all the arrows, writing $H^n(X, A)$ for the groups, $\delta : H^{n-1}(A) \to H^n(X, A)$ for the coboundary maps, and f^* for the induced maps.

HW#1 Write this definition out precisely, taking special care with the second and third axioms.

⑤ By combining the homology groups $H_n(X, A)$ for $n = 0, 1, 2, \ldots$ into a single graded group $H(X, A) = \oplus_{n=0}^{\infty} H_n(X, A)$ the exact sequence of (X, A) can be viewed as an exact triangle

$$\begin{array}{ccccc} & & H(X) & & \\ & \nearrow & & \searrow & \\ H(A) & & \xleftarrow[-1]{\quad\quad} & & H(X,A) \end{array}$$

with diagonal maps (induced by inclusions) of degree zero, and the horizontal boundary map of degree -1. Here a map of graded groups is said to be of degree k if it sends each element of pure degree i, meaning all its coordinates except the ith are zero, to an element of pure degree $i + k$.[†]

1.2 Exact Sequences

Exact sequences play a central role in algebraic topology. In this section we (re)acquaint the reader with some of their basic properties, and describe how to exploit these properties in homology calculations. We begin with a simple algebraic result that spells out some useful relationships among the groups and maps in any exact sequence of abelian groups.

Exact Sequence (ES) Lemma. ① *If* $\cdots \xrightarrow{p} A \xrightarrow{q} \cdots$ *is exact, then* $A = 0 \iff p = 0 = q$.

② *If* $\cdots \xrightarrow{p} A \xrightarrow{q} B \xrightarrow{r} \cdots$ *is exact, then* $q = 0 \iff p$ *is onto* $\iff r$ *is one-to-one, and* q *is an isomorphism* $\iff p$ *and* r *are zero. It follows (using ①) that every third group in a long exact sequence vanishes if and only if every third map is an isomorphism.*

③ *If* $\cdots \xrightarrow{p} A \xrightarrow{q} B \xrightarrow{r} C \xrightarrow{s} \cdots$ *is exact, then there is a short exact sequence*

$$0 \longrightarrow \operatorname{coker}(p) \longrightarrow B \longrightarrow \ker(s) \longrightarrow 0.$$

In particular, if C *is free abelian, then* $B \cong \operatorname{coker}(p) \oplus \ker(s)$ *by part* ⓑ *of the SES Lemma below.*

Proof ① and ② are straightforward. For ③, note that there is a short exact sequence

$$0 \longrightarrow \ker(r) \longrightarrow B \longrightarrow \operatorname{im}(r) \longrightarrow 0.$$

But $\ker(r) = \operatorname{im}(q) \cong A/\ker(q) = A/\operatorname{im}(p) = \operatorname{coker}(p)$ and $\operatorname{im}(r) = \ker(s)$. □

[†] Combining this perspective with some basic category theory (functors and natural transformations) the axioms can be rephrased succinctly. First some notation: For any admissible category TP of topological pairs (see §2 for the definition) let TP′ be its associated *homotopy category* in which the each morphism set $\operatorname{Map}((X, A), (Y, B))$ is replaced with $[(X, A), (Y, B)]$, and $R : \mathrm{TP}' \to \mathrm{TP}'$ be the restriction functor that sends any object (X, A) to A, and any map $(X, A) \to (Y, B)$ to its restriction $A \to B$. Also, let GG denote the category of all graded abelian groups and maps of degree zero. Then a homology theory on TP consists of a functor $H : \mathrm{TP}' \to \mathrm{GG}$ and a natural transformation $\partial : H \to H \circ R$ of degree -1 satisfying the last three Eilenberg-Steenrod axioms (where for *exactness* we use the exact triangle as above with bottom map $\partial_{X,A}$, *excision* remains the same, and *dimension* asserts that $H(\mathrm{pt})$ is concentrated in degree zero).

Now specialize to the case of a short exact sequence

$$(*) \qquad 0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

of abelian groups. Note that exactness at A means f is one-to-one, and exactness at C means g is onto. The sequence is said to split on the left if there exists a homomorphism $\ell : B \to A$ such that $\ell f = 1_A$, and to split on the right if there exists a homomorphism $r : C \to B$ such that $gr = 1_C$.† Here is a very useful result about splittings.

Short Exact Sequence (SES) Lemma. *The sequence* $(*)$ *splits on the left if and only if it splits on the right, in which case we just say that* $(*)$ *splits. Furthermore,* ⓐ *if* $(*)$ *splits, then* $B \cong A \oplus C$*;* ⓑ *if* C *is free abelian, then* $(*)$ *splits.*

Remarks 1) The sequence $0 \to \mathbb{Z} \xrightarrow{\times 2} \mathbb{Z} \longrightarrow \mathbb{Z}_2 \to 0$ is not split; do you see why?
2) The converse of ⓐ fails. For example, $0 \to \mathbb{Z} \to \mathbb{Z} \oplus \mathbb{Z}_2^\infty \to \mathbb{Z}_2^\infty \to 0$, where the first map sends n to $(2n, 0, 0, \dots)$ and the second sends $(n, k_1, k_2, \dots)$ to $(\overline{n}, k_1, k_2, \dots)$, is not split.

HW#2 Prove the SES lemma.

HW#3 Using the Eilenberg-Steenrod axioms, and the ES and SES Lemmas, prove:

(a) Homotopy equivalences are homology equivalences. Conclude that if X is contractible, then $H_n(X) \cong H_n(\text{pt})$ for all n.
(b) Retractions induce epimorphisms on homology. Furthermore, if A is a retract of X then
$$H_n(X) \cong H_n(A) \oplus H_n(X, A)$$
for all n (hint: use the exactness axiom), but this conclusion need not hold in general (use the example $X = B^1$, $A = S^0$, calling on the first result in (c) below to compute $H_n(S^0)$).
(c) If U and V are disjoint open subsets of a space X, then $H_n(U \cup V) \cong H_n(U) \oplus H_n(V)$ for each n. Deduce that if X has finitely many connected components $X_1, \dots, X_k$, then $H_n(X) \cong H_n(X_1) \oplus \cdots \oplus H_n(X_k)$. What if it has infinitely many components?

1.3 Diagram Chasing

This is a technique for studying commutative diagrams involving exact sequences. We describe it by means of the Short Five Lemma, whose proof involves a diagram chase, and then ask you to supply a proof of a useful generalization, the Five Lemma.

Short Five Lemma. *If the commutative diagram*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \xrightarrow{f} & B & \xrightarrow{g} & C & \longrightarrow & 0 \\ & & \downarrow p & & \downarrow q & & \downarrow r & & \\ 0 & \longrightarrow & D & \xrightarrow{h} & E & \xrightarrow{k} & F & \longrightarrow & 0 \end{array}$$

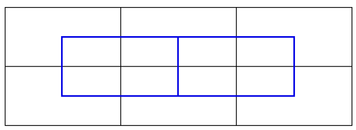
has exact rows, and if p *and* r *are isomorphisms, then* q *is an isomorphism.*

Proof To show q is 1-1, it suffices to show $\ker(q) = 0$. So given $b \in B$ with $q(b) = 0$, we must show $b = 0$. By commutativity of the right square $rg(b) = kq(b) = k(0) = 0$, and so $g(b) = 0$ since r is 1-1. Exactness at B shows $\exists a \in A$ with $f(a) = b$. Commutativity of the left square shows $hp(a) = qf(a) = q(b) = 0$. Thus $p(a) = 0$, since h is 1-1 (exactness at A'), and so $a = 0$ since p is 1-1. Thus $b = f(a) = f(0) = 0$, as desired.

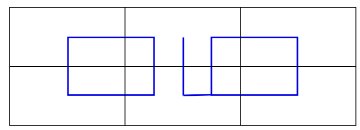
† Note that from set theory, g is onto $\iff$ g has a right inverse, i.e. $\exists$ a function r with $gr = 1$. So why doesn't every SES split on the right? The point is, we don't know that such an r can be found that is a homomorphism.

To show q is onto, pick any $e \in E$. Then $\exists c \in C$ with $r(c) = k(e)$, since r is onto, and so $\exists b \in B$ with $g(b) = c$, since g is onto (exactness at C). Commutativity of the right square shows $rg(b) = kq(b) = k(e)$, that is, k maps e and $e' := q(b)$ to the same element in F. So consider the difference $e'' = e - e'$ (this is a standard trick in diagram chasing arguments) which k maps to 0 since it is a homomorphism. By exactness at E, there exists $d \in D$ with $h(d) = e''$. Surjectivity of p then gives $a \in A$ with $p(a) = d$, so $e'' = hp(a) = qf(a)$. Thus $q(b + f(a)) = e' + e'' = e' + (e - e'') = e$. Therefore e is in the image of q, and so q is onto. ☐

Unfortunately, writing out the proof as we did above obscures the visual clarity of the technique. It is in fact a dynamic process of drawing arrows between dots (representing elements) in the diagram, many of which are best left unnamed. The following pictures provide the structure for such a proof, in the spirit of "proofs without words."



q is injective



q is surjective

HW#4 Prove the **Five Lemma**: *If the commutative diagram*

$$\begin{array}{ccccccccc} A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & D & \longrightarrow & E \\ \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle q} & & \downarrow{\scriptstyle r} & & \downarrow{\scriptstyle s} & & \downarrow{\scriptstyle t} \\ P & \longrightarrow & Q & \longrightarrow & R & \longrightarrow & S & \longrightarrow & T \end{array}$$

has exact rows, and if p, q, s, t *are isomorphisms, then* r *is an isomorphism.*

The Five Lemma is ubiquitous in homology theory calculations. For example it implies:

Homology Equivalence (HE) Lemma. *The following are equivalent for any triple* $A \subset B \subset X$:
- (a) *the inclusion* $A \hookrightarrow B$ *is a homology equivalence*
- (b) *the inclusion* $(X, A) \hookrightarrow (X, B)$ *is a homology equivalence*
- (c) $H_n(B, A) = 0$ *for all* n.

Proof (a)$\Longrightarrow$(b): The diagram

$$\begin{array}{ccccccccccccc} \cdots & \longrightarrow & H_n(A) & \longrightarrow & H_n(X) & \longrightarrow & H_n(X,A) & \longrightarrow & H_{n-1}(A) & \longrightarrow & H_{n-1}(X) & \longrightarrow & \cdots \\ & & \downarrow{\scriptstyle \cong} & & \downarrow{\scriptstyle \cong} & & \downarrow & & \downarrow{\scriptstyle \cong} & & \downarrow{\scriptstyle \cong} & & \\ \cdots & \longrightarrow & H_n(B) & \longrightarrow & H_n(X) & \longrightarrow & H_n(X,B) & \longrightarrow & H_{n-1}(B) & \longrightarrow & H_{n-1}(X) & \longrightarrow & \cdots \end{array}$$

commutes and has exact rows with vertical isomorphisms as indicated (explain why). Now apply the 5-lemma. The converse (b)$\Longrightarrow$(a) is proved in the same way, after a shift in the diagram. The equivalence (a)$\Longleftrightarrow$(c) follows from the exact sequence of the pair (B, A), using the ES Lemma. ☐

The HE Lemma can be used to establish variants of the excision axiom, which asserts that U can be excised from (X, A) (i.e. the inclusion $(X - U, A - U) \hookrightarrow (X, A)$ is a homology equivalence) provided $\overline{U} \subset A^\circ$. Can this condition on U be relaxed? For example, can one always excise A° from (X, A)? The answer is "no" in general, but "yes" if A is "nicely" embedded in X. To make this precise we recall Hatcher's notion of a "good pair" (also called an "NDR-pair" in the literature, standing for "neighborhood deformation retract").

Definition The pair (X, A) is called a good pair if A is closed in X and is a strong deformation retract of some neighborhood N of itself, denoted $N \searrow A$.

By definition, this means there exists a retraction $r : N \to A$ with $i \circ r \simeq \mathbb{1}_N$ (rel A), where $i : A \hookrightarrow N$ is the inclusion. (Recall that r being a retraction just means $r \circ i = \mathbb{1}_A$.) We say that (X, A) is very good if A is closed in X and $(B, \partial B)$ is good, where $B = X - A^\circ$ and ∂B is the "boundary" or "frontier" of B, i.e. $\partial B = A \cap B = \partial A$.

HW#5 Show that any very good pair (X, A) is good. In particular, show that if $N \searrow \partial B$ in $B = X - \operatorname{int} A$, then $A \cup N \searrow A$ in X.

Now if (X, A) is very good, with $B = X - A^\circ$ and $N \searrow \partial B$ in B as above, then both $\partial B \hookrightarrow N$ and $A \hookrightarrow A \cup N$ are homology equivalences. It follows from the HE-lemma that the vertical maps in the following commutative diagram

$$\begin{array}{ccc} H_n(B, \partial B) & \longrightarrow & H_n(X, A) \\ \downarrow & & \downarrow \\ H_n(B, N) & \longrightarrow & H_n(X, A \cup N) \end{array}$$

are isomorphisms. The lower horizontal map is also an isomorphism, by excision, and so the upper one is as well. This proves:

Excision Lemma. *If (X, A) is very good, then A° can be excised from (X, A).*

For example, (S^k, B) is very good for any closed hemisphere B of S^k, so excising B° from (S^k, B) yields an isomorphism $H_n(S^k, B) \cong H_n(B^k, S^{k-1})$ since $S^k - B^\circ$ is clearly homeomorphic to B^k.

HW#6 Using this observation, show that $H_n(S^k) \cong H_{n-1}(S^{k-1})$ for any $n > 1$ and $k \geqslant 1$. Conclude that $H_n(S^k) = 0$ for all $n > k \geqslant 1$. What goes wrong when $n = 1$?

Similarly if a manifold M is the union $P \cup Q$ of two codimension zero submanifolds that meet along their common boundary, then Q° can be excised from (M, Q) (which is a very good pair by Morton Brown's "collaring theorem"†) giving $H_n(M, Q) \cong H_n(P, \partial P)$.

The HE-lemma can also be used to elucidate the relationship between *relative* and *absolute* homology, in particular between $H_n(X, A)$ and $H_n(X/A)$, where X/A denotes the quotient space of X with A collapsed to a point:

Relative Homology (RH) Lemma. *If (X, A) is good, then $H_n(X, A) \cong H_n(X/A, A/A)$ for all q. This last group is isomorphic to the "reduced homology" $\widetilde{H}_n(X/A)$ of X/A, defined in the next section. Thus for good pairs (X, A), we have $H_n(X, A) = \widetilde{H}_n(X/A)$.*

Proof For notational efficiency we write $\widehat{Y} = Y/A$ and $Y_0 = Y - A$ for any $Y \supset A$. Then there is a commutative diagram

$$\begin{array}{ccccc} H_n(X, A) & \longrightarrow & H_n(X, N) & \longleftarrow & H_n(X_0, N_0) \\ \downarrow & & \downarrow & & \downarrow \\ H_n(\widehat{X}, \widehat{A}) & \longrightarrow & H_n(\widehat{X}, \widehat{N}) & \longleftarrow & H_n(\widehat{X}_0, \widehat{N}_0) \end{array}$$

with vertical maps induced by the natural projection $p : X \to \widehat{X}$ and horizontal maps induced by inclusions. Now the two horizontal maps on the left are isomorphisms by the HE Lemma (one must check that $N \searrow A$ in $X \Longrightarrow \hat{N} \searrow \hat{A}$ in $\widehat{X}$) as are the ones on the right (they are excisions). So is the rightmost vertical map, because $p|X_0 : X_0 \to \widehat{X}_0$ is a homeomorphism. It follows readily that the other two vertical maps are also isomorphisms, the leftmost one being the desired one. □

† A simple proof due to R. Conelly is given in Appendix B of Vick's *Homology Theory*

1.4 Reduced homology

Let (H, ∂) be a homology theory with coefficient group $\mathbb{Z}$. To avoid having to make special arguments about long exact sequences at the H_0 level (which arise since $H_0(\mathrm{pt}) \neq 0$) it is convenient to have a "reduced" homology theory $\widetilde{H}$ for which $\widetilde{H}_0(\mathrm{pt}) = 0$. So define

$$\widetilde{H}_n(X) = \ker(H_n(X) \to H_n(\mathrm{pt}))$$

where the map on the right is induced by the constant map. If $A \subset X$ is nonempty, set

$$\widetilde{H}_n(X, A) = H_n(X, A)$$

or equivalently $\widetilde{H}_n(X, A) = \ker((H_n(X, A) \to H_n(\mathrm{pt}, \mathrm{pt}))$ since the last group is trivial. Thus the reduced homology of (X, A) is identical to its regular homology except when $n = 0$ and $A = \varnothing$, in which case it is a proper subgroup of $H_0(X)$ satisfying

$$H_0(X) \cong \widetilde{H}_0(X) \oplus \mathbb{Z}$$

by the SES Lemma, since the short exact sequence $0 \to \widetilde{H}_0(X) \hookrightarrow H_0(X) \to H_0(\mathrm{pt}) = \mathbb{Z} \to 0$ splits. In particular $\widetilde{H}_n(\mathrm{pt}) = 0$ for *all* n.

HW#7 Show that reduced homology (with suitably defined boundary maps and induced maps) satisfies all the Eilenberg-Steenrod axioms except excision. For the exactness use the following general result, proved using a diagram chase:

Proposition. *The kernel $(K, \partial|)$ of any chain map $f\colon (C, \partial) \to (C', \partial')$ between chain complexes is a chain complex. If both (C, ∂) and $(C'\partial')$ are exact and f is onto, then $(K, \partial|)$ is also exact.*

Here $K = \ker(f\colon C \to C')$ and $\partial|$ is the restriction of ∂, which must be shown to map K to K.

HW#8 Prove by induction on n that $\widetilde{H}_n(S^k) = \mathbb{Z}$ for $n = k$ and vanishes for all other n.[†]
Conclude that for $k > 0$

$$H_n(S^k) \;=\; \begin{cases} \mathbb{Z} & \text{if } n = 0 \text{ or } k \\ 0 & \text{otherwise} \end{cases}$$

(Note that from HW#3b), $H_0(S^0) = \mathbb{Z} \oplus \mathbb{Z}$ and $H_n(S^0) = 0$ for $n > 0$.)

1.5 Other computational tools

In this section we construct two exact sequences that are useful for computing homology groups, the Mayer-Vietoris Sequence and the Sequence of a Triple.

Mayer-Vietoris Sequence. *For any triad $(X; A, B)$ (meaning $X = A \cup B$) such that A and B are either both open in X, or both closed in X with at least one of (X, A) or (X, B) very good, there is an exact sequence*

$$\cdots \xrightarrow{\Delta} H_{n+1}(X) \xrightarrow{\Delta} H_n(A \cap B) \xrightarrow{(\alpha_*, \beta_*)} H_n(A) \oplus H_n(B) \xrightarrow{a_* - b_*} H_n(X) \xrightarrow{\Delta} \cdots$$

where $A \xleftarrow{\alpha} A \cap B \xrightarrow{\beta} B$ and $A \xrightarrow{a} X \xleftarrow{b} B$ are inclusions, and Δ is the composition

$$H_n(X) \longrightarrow H_n(X, A) \xleftarrow{\cong} H_n(B, A \cap B) \xrightarrow{\partial} H_{n-1}(A \cap B)$$

where the middle map is excision. There is a completely analogous version of this sequence for reduced homology (just add tildes).

† Hint: Consider the pairs (S^{k+1}, B), where B is the southern hemisphere, and (B^{k+1}, S^k).

Leopold Vietoris
on his 110th birthday

Exercise Rewrite the Mayer-Vietoris sequence as an exact triangle.

Remark This sequence can be described geometrically: On the nth level, the maps are induced by inclusions. The first sends $\overline{\alpha}$ (for any n-cycle $\alpha \subset A \cap B$) to $(\overline{\alpha}, \overline{\alpha})$ (viewing α as a cycle in A and B, resp.), while the second sends $(\overline{\alpha}, \overline{\beta})$ to $\overline{\alpha - \beta}$. The boundary map Δ sends $\overline{\alpha}$ to $\overline{\partial}$, where $\alpha = \alpha_A \cup \alpha_B$, meeting along their common boundary ∂, with $\alpha_A \subset A$ and $\alpha_B \subset B$.

The proof of the Mayer-Vietoris Sequence relies on the following algebraic lemma:

Barrett-Whitehead Lemma. *Given a commutative diagram*

$$\begin{array}{ccccccccccccc}
\cdots & \longrightarrow & C_{n+1} & \xrightarrow{\partial} & A_n & \xrightarrow{f} & B_n & \xrightarrow{g} & C_n & \xrightarrow{\partial} & A_{n-1} & \longrightarrow & \cdots \\
 & & \downarrow{\cong}\, r & & \downarrow p & & \downarrow q & & \downarrow{\cong}\, r & & \downarrow p & & \\
\cdots & \longrightarrow & C'_{n+1} & \xrightarrow{\partial'} & A'_n & \xrightarrow{f'} & B'_n & \xrightarrow{g'} & C'_n & \xrightarrow{\partial'} & A_{n-1} & \longrightarrow & \cdots
\end{array}$$

with exact rows, where all the r maps are isomorphisms, there exists a long exact sequence

$$\cdots \longrightarrow B'_{n+1} \xrightarrow{\Delta} A_n \xrightarrow{(p,f)} A'_n \oplus B_n \xrightarrow{f'-q} B'_n \xrightarrow{\Delta} A_{n-1} \longrightarrow \cdots$$

where $\Delta = \partial \circ r^{-1} \circ g'$.

Exercises 1) Prove the lemma by a diagram chase.

2) Deduce the Mayer-Vietoris Sequence (and its reduced version) by applying the Barrett-Whitehead Lemma with the upper and lower sequences equal to the exact sequences of the pairs $(B, A \cap B)$ and (X, A), respectively.

3) "Prove" the Mayer-Vietoris Sequence from its geometric description in the Remark above.

HW#9 Use the Mayer-Vietoris Sequence to calculate

a) $H_n(S^k)$ (again)

b) $H_n(T^k)$, where T^k is the k-torus (the product of k copies of the circle)

for all n and k.

HW#10 Let M and N be manifolds, and $M \vee N$ be their *wedge product*, the quotient space of the disjoint union $M \sqcup N$ obtained by identifying a point in M with a point in N. Use the Mayer-Vietoris Sequence to calculate the homology of $M \vee N$. Then use this in turn to compute the homology of the pair (S^k, S^{k-1}) (viewing $S^{k-1} \subset S^k$ by identifying $\mathbb{R}^k$ with $\mathbb{R}^k \times \{0\}$ in $\mathbb{R}^{k+1}$), or equivalently the reduced homology of the quotient space $S^k/S^{k-1} \cong S^k \vee S^k$ (see the RH Lemma).

Exact Sequence of a Triple. *For any triple $A \supset B \supset C$ of spaces, there is an exact sequence*

$$\cdots \longrightarrow H_{n+1}(A,B) \xrightarrow{\Delta} H_n(B,C) \longrightarrow H_n(A,C) \longrightarrow H_n(A,B) \xrightarrow{\Delta} \cdots$$

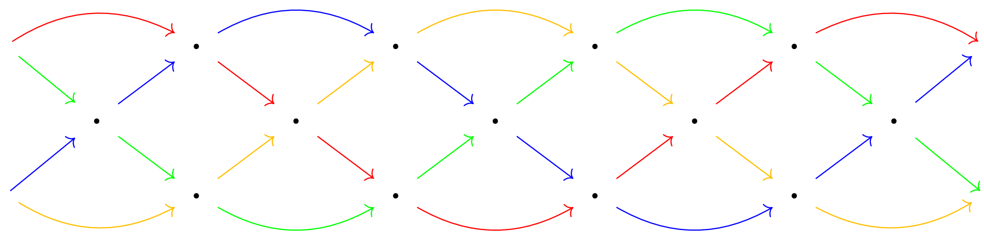
where the unlabeled maps are induced by inclusions, and Δ is the composition of the boundary map $H_{n+1}(A,B) \xrightarrow{\partial} H_n(B)$ with the map $H_n(B) \to H_n(B,C)$ induced by inclusion.

As with the Mayer-Vietoris Sequence, this can be proved using a suitable algebraic lemma. To state it, recall that a *chain complex* is a sequence of abelian groups joined by homomorphisms

$$\cdots \xrightarrow{\partial} C_{n+1} \xrightarrow{\partial} C_n \xrightarrow{\partial} C_{n-1} \xrightarrow{\partial} \cdots$$

such that any composition of two consecutive maps in the sequence is zero.

Braid Lemma. *If three of the four braided chain complexes in the following commutative diagram (distinguished by their colors) are exact, then so is the fourth:*



<u>Exercises</u> 1) Prove the lemma by a diagram chase.

2) Deduce the Sequence of a Triple by applying the Braid Lemma. (Hint: Fill in the red, blue and green strands with the sequences of the pairs (A, C), (A, B) and (B, C), respectively, and observe that this makes the yellow strand into the desired sequence of the triple (A, B, C). The only nontrivial thing to check is that the composition $H_n(B, C) \to H_n(A, C) \to H_n(A, B)$ is zero, which follows from the axioms since this composition also factors as $H_n(B, C) \to H_n(B, B) \to H_n(A, B)$ (by functoriality) and $H_n(B, B) = 0$ (by the exact sequence of (B, B)).)

HW#11 Use the Sequence of a Triple and induction to calculate $H_n(S^{k+\ell}/S^k)$ for all n, k and ℓ. (A simpler way to make this calculation is to note that $S^{k+\ell}/S^k$ is homotopy equivalent to a wedge of spheres; do you see why?)

HW#12 If you feel inspired, complete the exercises on the last two pages.

For later use, we take this opportunity to reformulate the definitions of chain complexes and chain maps (see page 3) in terms of graded groups:

<u>Definition</u> A <u>chain</u> <u>complex</u> is a graded group $C = \oplus_n C_n$ equipped with a <u>boundary map</u> $\partial \colon C \to C$, meaning an endomorphism of degree -1 for which $\partial^2 = 0$. We denote this chain complex by the pair (C, ∂), where the grading is implicit. A <u>chain</u> <u>map</u> $f \colon (C, \partial) \to (C', \partial')$ between chain complexes is a map of degree 0 that commutes with the boundary maps, i.e. $f\partial' = \partial f$.

For the record, we reiterate that such a chain map is really a sequence of maps $f_n : C_n \to C'_n$

$$\begin{array}{ccccccccc}
\cdots & \longrightarrow & C_{n+1} & \xrightarrow{\partial_{n+1}} & C_n & \xrightarrow{\partial_n} & C_{n-1} & \longrightarrow & \cdots \\
 & & \downarrow{\scriptstyle f_{n+1}} & & \downarrow{\scriptstyle f_n} & & \downarrow{\scriptstyle f_{n-1}} & & \\
\cdots & \longrightarrow & C'_{n+1} & \xrightarrow{\partial'_{n+1}} & C'_n & \xrightarrow{\partial'_n} & C'_{n-1} & \longrightarrow & \cdots
\end{array}$$

that make the diagram commute: $\partial'_n f_n = f_{n-1}\partial_n$ for all n. The graded formulation is just a more efficient way to say this.

2. Constructing Homology Theories

Every homology theory is constructed as a composition of two functors:

$$\text{TP} \longrightarrow \text{CC} \longrightarrow \text{GG}$$

Here TP is an "admissible" category of topological pairs (defined below; it need not be the full category TOP of all topological pairs) while CC and GG are the algebraic categories of chain complexes and graded abelian groups, respectively. So the first topological step TP → CC (which is different for each theory) passes from topology to algebra, while the second algebraic step CC → GG (which is the same for all theories) stays within algebra.

To define TP more precisely, we use the following terminology: A singleton is any space consisting of a single point. Any map from a singleton into a nonempty space is called a trivial map. The product inclusions of a pair (X, A) are the maps $(X, A) \to (X \times [0,1], A \times [0,1]))$ given by $x \mapsto (x, 0)$ and $x \mapsto (x, 1)$. The lattice of a pair (X, A) is the category consisting of all 6 pairs arising from the triple $(X, A, \varnothing)$ and all 36 inclusions between such pairs. The lattice of a map $f : (X, A) \to (Y, B)$ of pairs is the category consisting of the lattices of (X, A) and (Y, B), and all the maps induced by f between corresponding pairs in those lattices. In general, we say that a subcategory $\mathcal{A}$ of TOP contains a pair (X, A), a map of pairs f, or another subcategory $\mathcal{B}$, to mean that (X, a) is an object in $\mathcal{A}$, f is a morphism in $\mathcal{A}$, or $\mathcal{B}$ is a subcategory of $\mathcal{A}$. We can now describe the allowable topological categories TP :

Definition (Eilenberg and Steenrod, 1952) An admissible category of topological pairs is a subcategory TP of TOP that contains all trivial maps, all product inclusions of objects in TP, and all lattices of morphisms in TP.

As for the algebraic categories, the objects in CC are chain complexes of abelian groups and the morphisms are chain maps between them (see pages 3 and 11). The objects in GG are $\mathbb{Z}$-graded abelian groups (or equivalently sequences of abelian groups) and the morphisms are degree zero maps between them (or equivalently maps between the corresponding groups in the sequences).

2.1 The Algebraic Step

There is a standard functor $H : \text{CC} \to \text{GG}$ that is the same for all homology theories. It sends any chain complex (C, ∂) in CC to its graded homology group

$$H = H(C, \partial) \ := \ Z/B$$

where $Z = \ker(\partial)$ and $B = \operatorname{im}(\partial)$ (referred to as respectively as the cycles and boundaries in (C, ∂)). Note that $B \subset Z$ since $\partial^2 = 0$, so this quotient makes sense.† Thus each element of H is a coset $z + B$ of B in Z, often denoted $\overline{z}$ and referred to as the homology class of the cycle z.

To see how H is graded, recall that this chain complex (C, ∂) can be viewed as a sequence

$$\cdots \xrightarrow{\partial_{n+2}} C_{n+1} \xrightarrow{\partial_{n+1}} C_n \xrightarrow{\partial_n} C_{n-1} \xrightarrow{\partial_{n-1}} \cdots$$

with each $\partial_n \partial_{n+1} = 0$. Set $Z_n = \ker(\partial_n)$ (the n-cycles) and $B_n = \operatorname{im}(\partial_{n+1})$ (the n-boundaries), and note that $B_n \subset Z_n$. The nth homology group of the complex is the quotient $H_n = Z_n/B_n$, which measures the inexactness of (C, ∂) at C_n and gives the graded structure $H = \oplus_n H_n$.

† Also note that a chain complex (C, ∂) is exact if and only if its homology $H(C, \partial)$ is trivial.

As for morphisms, H sends each chain map $f : (C, \partial) \to (C', \partial')$ to the naturally induced map $f_* : H(C, \partial) \to H(C', \partial')$ of graded groups defined by $f_*(\overline{z}) = \overline{f(z)}$.

HW#13 Show that this map f_* is a well defined homomorphism of graded groups of degree zero (hint: Use the universal property of quotient groups) and that $\mathbb{1}_* = \mathbb{1}$ and $(fg)_* = f_* g_*$ (when the composition is defined).

How to actually compute the homology of a chain complex

$$(C, \partial) \;:\quad \cdots \xrightarrow{\partial_{n+2}} C_{n+1} \xrightarrow{\partial_{n+1}} C_n \xrightarrow{\partial_n} C_{n-1} \xrightarrow{\partial_{n-1}} \cdots$$

This reduces to computing each $H_n = H_n(C, \partial) = Z_n/B_n = \ker(\partial_n)/\mathrm{im}(\partial_{n+1})$, which in turn depends only on the chain group C_n and the incoming and outgoing boundary maps ∂_{n+1} and ∂_n. We show how to carry this out when each C_n is finitely generated and free abelian. But first:

A quick "review" of linear algebra over $\mathbb{Z}$ (from Father Guido Sarducci's 5-Minute University?)

Let A be a finitely generated abelian group. Then there is an isomorphism

$$A \;\cong\; \mathbb{Z}^r \oplus \mathbb{Z}_{a_1} \oplus \cdots \oplus \mathbb{Z}_{a_t}$$

for a unique nonnegative integer r (the rank of A, written $\mathrm{rk}(A)$) and some list $\vec{a} = (a_1, \ldots, a_t)$ of positive integers (a torsion list for A). Note that $\vec{a}$ is not unique.†

The group A is free abelian iff $\vec{a}$ is empty or consists of 1s only, so iff $A \cong \mathbb{Z}^r$ for some r. It is a nontrivial fact (not proved here) that any subgroup B of A is then also free abelian of $\mathrm{rk}(B) \leqslant \mathrm{rk}(A)$. Furthermore, the quotient group A/B can be computed by diagonalizing the inclusion map $B \hookrightarrow A$, as explained in a more general setting below:

Any homomorphism $\mathbb{Z}^q \to \mathbb{Z}^r$ between finite rank free abelian groups – given by an $r \times q$ integer matrix M with respect to the standard bases – can be diagonalized. This means we can change bases to reduce M to a positive diagonal matrix $D = (d_{ij})$, meaning its entries $d_{11}, \ldots, d_{tt}$ are all positive for some $t \geqslant 0$, and all other entries of D are zero. Such a matrix D (which is not unique) is called a Smith Form of M, written $D = \mathrm{SF}(M)$. It is specified by its list

$$\mathrm{tor}(M) \;:=\; (d_{11}, \ldots, d_{tt})$$

of nonzero diagonal, which we call a torsion list for M. This list is all we need to compute homology.

We now show how to extract $\mathrm{tor}(M)$ from M using a single operation OP = add or subtract one row or column of M from another. Initially set $\mathrm{tor}(M) = ()$, the empty list. If $M = 0$ then you're done. Otherwise, inductively expand $\mathrm{tor}(M)$ as follows: Let m be a nonzero entry in M of smallest absolute value, and set $|M| = |m|$. If m divides all the other entries in its row $\vec{r}$ and column $\vec{c}$, then 1) append $|m|$ to the list $\mathrm{tor}(M)$, and 2) shrink M (using OP to zero out all the other entries in $\vec{r}$ and $\vec{c}$, then delete $\vec{r}$ and $\vec{c}$) and induct on the size of M. If m does not divide some entry in $\vec{r}$ or $\vec{c}$, then use OP to replace M with a matrix with smaller $|M|$, and again induct.

Although not unique, $\mathrm{tor}(M) = (a_1, \ldots, a_t)$ gives important information about M. Indeed it is geometrically evident that $\mathrm{im}(M) \cong \mathbb{Z}^t$, $\ker(M) \cong \mathbb{Z}^{q-t}$, and

$$\mathrm{coker}(M) \;=\; \mathbb{Z}^r/\mathrm{im}(M) \cong \mathbb{Z}^{r-t} \oplus \mathbb{Z}_{a_1} \oplus \cdots \oplus \mathbb{Z}_{a_t}.$$

Note $\mathrm{rk}(M) := \mathrm{rk}(\mathrm{im}(M)) = t$ and $\mathrm{nul}(M) := \mathrm{rk}(\ker(M)) = q - t$, which implies the rank plus nullity theorem $q = \mathrm{rk}(M) + \mathrm{nul}(M)$ in this setting.

† Since $\mathbb{Z}_a \oplus \mathbb{Z}_b \cong \mathbb{Z}_{ab}$ iff $\gcd(a, b) = 1$, however, we can arrange for $a_1 \leqslant \cdots \leqslant a_t$, with each a_i a divisor of a_{i+1}, or each a_i a prime power. With either modification, $\vec{a}$ *is* unique, giving the *invariant* or *primary* forms of the classification of finitely generated abelian groups. Note: the reason for the terminology "torsion list" is that the torsion elements in A (meaning the elements of finite order) form a subgroup $\mathrm{Tor}(A)$ isomorphic to $\mathbb{Z}_{a_1} \oplus \cdots \oplus \mathbb{Z}_{a_t}$.

Now to compute H_n, note that the short exact sequence $0 \to Z_n \hookrightarrow C_n \xrightarrow{\partial_n} B_{n-1} \to 0$ splits since B_{n-1} (as a subgroup of the free abelian group C_n) is free abelian. Thus $C_n \cong Z_n \oplus B_{n-1}$, so

$$\operatorname{coker}(\partial_{n+1}) = C_n/B_n \;\cong\; (Z_n/B_n) \oplus B_{n-1} = H_n \oplus \operatorname{im}(\partial_n).$$

By the green remarks, $\operatorname{coker}(\partial_{n+1}) \cong \mathbb{Z}^{c_n - r_{n+1}} \oplus \mathbb{Z}_{a_1} \oplus \cdots \oplus \mathbb{Z}_{a_t}$ and $\operatorname{im}(\partial_n) \cong \mathbb{Z}^{r_n}$, where $c_n = \operatorname{rk}(C_n)$, $r_i = \operatorname{rk}(\partial_i)$, and $(a_1, \ldots, a_t)$ is any torsion list for ∂_{n+1}. It follows that

$$(*) \qquad H_n \;\cong\; \mathbb{Z}^{c_n - (r_{n+1} + r_n)} \oplus \mathbb{Z}_{a_1} \oplus \cdots \oplus \mathbb{Z}_{a_t}.$$

Summarizing: All we need to compute the homology of a chain complex (C, ∂) whose chain groups are all free abelian of finite rank are the torsion lists $\operatorname{tor}(\partial_n)$ of the boundary maps![†]

HW#14 Compute a torsion list for the 7×5 matrix M whose entries in its first four rows (reading left to right / top to bottom) are the first 20 positive integer primes, and whose remaining entries are zero. Then use this to compute the homology of any short chain complex $\mathbb{Z}^5 \xrightarrow{M} \mathbb{Z}^7 \xrightarrow{L} \mathbb{Z}^r$ where L is an integer matrix of rank 2 with $LM = 0$.

2.2 The Topological Step

We turn now to the construction of functors TP $\to$ CC. In particular, we will describe three such functors, defined on three increasingly large admissible categories: SC = simplicial complexes, CW = cell complexes (a.k.a. CW-complexes), and finally TOP = all topological pairs.

To explain the basic construction, we first informally describe what a simplicial complex X is (ignoring orientation issues that complicate the picture) and define an associated chain complex $(C(X), \partial)$ of $\mathbb{Z}_2$-modules (= vector spaces over $\mathbb{Z}_2$) rather than $\mathbb{Z}$-modules (= abelian groups) since we are ignoring orientations. After that, we give a more precise treatment for all three categories SC, CW, and TOP, and incorporating orientations to work over $\mathbb{Z}$.

Here's the basic idea: An n-dimensional simplex (or just n-simplex) is any subset of a euclidean space that is the convex hull of $n+1$ "independent" points, meaning the points do not lie in any affine space of dimension $< n$. Thus simplices of dimension 0, 1, 2 and 3 are just points, closed intervals, solid triangles, and solid tetrahedra.

A simplicial complex is a union X of simplices in some euclidean space, any two of which intersect in a common face. Any finite formal sum (= $\mathbb{Z}_2$-linear combination) of the n-simplices in X is called a (mod 2) n-chain in X. The set $C_n(X; \mathbb{Z}_2)$ of *all* such chains forms a $\mathbb{Z}_2$-vector space, where addition is defined in the obvious way, and the zero element is just the empty linear combination. The n-simplices in X form a natural basis for $C_n(X; \mathbb{Z}_2)$, so all the simplices in X form a natural basis for the graded $\mathbb{Z}_2$-vector space $C(X; \mathbb{Z}_2) = \oplus_n C_n(X; \mathbb{Z}_2)$.

There is a linear "boundary" map $\partial\colon C(X; \mathbb{Z}_2) \to C(X; \mathbb{Z}_2)$ of degree -1 (i.e. a sequence of linear maps $\partial_n : C_n(X; \mathbb{Z}_2) \to C_{n-1}(X; \mathbb{Z}_2)$) defined on each simplex to be the sum of all its $(n-1)$-dimensional faces (the two endpoints of an interval, three edges of a triangle, four triangular faces of a tetrahedron, etc). An easy check shows $\partial^2 = 0$ (i.e. $\partial_n \partial_{n+1} = 0$ for every n; see HW#17 below). The result is the $\mathbb{Z}_2$-simplicial chain complex $(C(X; \mathbb{Z}_2), \partial)$:

$$\longrightarrow \; C_{n+1}(X) \; \xrightarrow{\partial_{n+1}} \; C_n(X) \; \xrightarrow{\partial_n} \; C_{n-1}(X) \; \longrightarrow \; \cdots$$

whose homology $H(C(X; \mathbb{Z}_2), \partial)$, also denoted $H(X; \mathbb{Z}_2)$, is called the mod 2 homology of X. Subscripts can be added to denote the graded components, e.g. $H_n(X; \mathbb{Z}_2) = \ker \delta_n / \operatorname{im} \delta_{n+1}$.

[†] If we work over a field F (with chain complexes of vector spaces over F rather than abelian groups) then noting that linear maps between vector spaces have trivial torsion, $(*)$ becomes $H_n(C, \partial; F) \cong F^{c_n - (r_{n+1} + r_n)}$.

Example The 2-simplex (solid triangle) Δ, viewed as the union of all its faces, has three vertices, three edges, and one face (the triangle itself), has $\mathbb{Z}_2$-simplicial chain complex

$$0 \to \mathbb{Z} \xrightarrow{\partial_2} \mathbb{Z}^3 \xrightarrow{\partial_1} \mathbb{Z}^3 \to 0 \qquad \text{where} \qquad \partial_1 = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \quad \text{and} \quad \partial_2 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

(The reader should verify that $\partial_1\partial_2 = 0$).

Now for a more precise treatment:

We begin by describing the *absolute* objects in the various categories – i.e. a single simplicial complex, cell complex, or topological space X, rather than a pair (X, A) – and the morphisms between them – called simplicial maps, cellular maps, or just (continuous) maps, respectively. Next we define the associated chain complexes $(C(X), \partial)$. And at the end of the section, we address the relative case, showing algebraically how to use $C(X)$ and $C(A)$ to define the relative chain complexes $(C(X, A), \partial)$ when $A \neq \varnothing$.

Simplicial Complexes SC

Simplices

Start with $n+1$ points $x_0, \dots, x_n$ in a euclidean space. Their convex hull is the set

$$[x_0, \dots, x_n] \;=\; \Big\{ \sum_{i=0}^{n} t_i x_i \;\Big|\; t_i \geqslant 0,\; \sum_{i=0}^{n} t_i = 1 \Big\}.$$

If $x_0, \dots, x_n$ are independent, meaning they do not lie in any affine subspace of dimension $< n$, then $[x_0, \dots, x_n]$ is called an n-simplex, with vertices $x_0, \dots, x_n$. Thus the vertices of a 1-simplex are distinct, those of a 2-simplex are noncollinear, those of a 3-simplex are noncoplanar, etc.

HW#15 Show that the points $x_0, \dots, x_n$ are independent $\iff$ the vectors $x_1 - x_0, \ \dots \ , x_n - x_0$ are linearly independent.

The coefficients $(t_0, \dots, t_n)$ for any point $x = \sum_{i=0}^n t_i x_i$ in a simplex $\sigma = [x_0, \dots, x_n]$ are unique, and are called the barycentric coordinates of x. For example the vertices of σ are the points with one barycentric coordinate equal to 1 (and the rest 0), and it follows that they are uniquely determined by σ, characterized geometrically as the points in σ that don't lie in an open interval contained entirely in σ. Thus there are exactly $n!$ distinct ways to write σ in the form $[x_0, \dots, x_n]$, corresponding to the $n!$ ways to reorder its vertices $x_0, \dots, x_n$. For example, a 1-simplex $[x_0, x_1] = [x_1, x_0]$, and a 2-simplex

$$[x_0, x_1, x_2] = [x_1, x_2, x_0] = [x_2, x_0, x_1] = [x_2, x_1, x_0] = [x_1, x_0, x_2] = [x_0, x_2, x_1].$$

A simplex equipped with an ordering of its vertices is called an ordered simplex.

The interior of a simplex σ is called an open simplex. It is the set of all points in σ whose barycentric coordinates are *all* nonzero, while the boundary $\partial\sigma$ of σ is the complement of its interior. The barycenter b_σ of σ is the unique point whose barycentric coordinates are all equal, so $b_\sigma = (1/(n+1), \dots, 1/(n+1))$ if σ is an n-simplex.

Faces of a simplex

Each n-simplex has 2^n faces, where a face just means a simplex spanned by a subset of the vertices. In particular, $\sigma = [x_0, \dots, x_n]$ has $n+1$ *codimension one* faces

$$\sigma_i \;:=\; [x_0, \dots, \hat{x}_i, \dots, x_n] \qquad \text{for } i = 0, \dots, n$$

where the caret over x_i indicates that it should be omitted (note that this notation depends on a choice of ordering of the vertices).

Simplicial complexes and simplicial maps

Strictly speaking, we'll define *finite* simplicial complexes and *finite* cell complexes. With a little more work, one can also define *infinite* ones, but as we limit our discussion to finite ones, we drop the prefix *finite*.

Definition A simplicial complex X is any finite set of simplices in a euclidean space for which a) any face of a simplex in X is another simplex in X, and b) any two simplices in X intersect in a common face.

The union of all the simplices in X is denoted $|X|$ (although by abuse of notation, we sometimes simply write X for $|X|$, although this is an abuse of notation since the simplex decomposition is part of the structure). A simplicial structure or triangulation of a topological space Y is a simplicial complex X together with a homeomorphism $|X| \to Y$. Note that any simplicial complex is the *disjoint* union of its open simplices.

Definition A simplicial map $f\colon X \to Y$ between simplicial complexes X and Y is a (continuous) map $f\colon |X| \to |Y|$ that sends each simplex in X affinely onto a simplex in Y.

In particular, f must map vertices in X to vertices in Y, but in general it *need not* preserve the dimension of every simplex. For example, it could map all of X onto a single vertex of Y.

Oriented simplices

An oriented simplex is a simplex equipped with an *equivalence class* of orderings of its vertices, where the equivalence is generated by *even* permutations. Thus there are *exactly two* oriented simplices with a given set of vertices; if σ is one such, then $-\sigma$ will denote the other. For example the six 2-simplices displayed above are distinct as ordered simplices, whereas the first three represent one oriented 2-simplex σ, and the last three represent $-\sigma$.

The standard oriented n-simplex is $\Delta^n = [e_0, \ldots, e_n] \subset \mathbb{R}^n$, where e_0 is the origin, and $e_1, \ldots, e_n$ are the tips of the standard basis vectors in $\mathbb{R}^n$.

The boundary of an oriented simplex:

The boundary $\partial\sigma$ of an oriented simplex σ is the union $\sigma_0 \cup \cdots \cup \sigma_n$ of its codimension one faces, or more precisely, their *signed* sum:

$$\partial\sigma \;=\; \sum_{i=0}^{n}(-1)^i\sigma_i \;=\; \sum_{i=0}^{n}(-1)^i[x_0, \ldots, \hat{x}_i, \ldots, x_n].$$

Thus $\partial\sigma$ is an integer linear combination of oriented simplices. A priori, this definition appears to depend on an ordering of the vertices, but in fact it does not:

HW#16 Show that $\partial\sigma$ is well-defined for *oriented* n-simplices σ, i.e. unchanged as a sum of oriented simplices under any even permutation of the vertices of σ. (Hint: Show that $\partial\sigma$ changes sign if we transpose any two adjacent vertices.)

The key to constructing a chain complex $C(X), \partial)$ from a simplicial complex X is the observation that the boundary of the boundary of any simplex is zero. We leave the proof of this observation to the reader:

HW#17 Show that $\partial^2\sigma = 0$. (Hint: The simplex obtained from $[x_0, \ldots, x_n]$ by omitting the ith and jth vertices, for $i < j$, is equal to $(\sigma_i)_{j-1}$ and also to $(\sigma_j)_i$.)

Before moving on to the construction of $C(X), \partial)$, we introduce the relevant features of other two categories CW and TOP.

Cell Complexes CW

Cells: An n-cell is (a copy of) the closed n-ball B^n, equipped with its natural orientation inherited from $\mathbb{R}^n$, and an open n-cell is the interior of an n-cell. A cell just means an n-cell for some n.

Cell complexes and cellular maps:

Definition A cell complex or CW complex is a space X equipped with a filtration

$$X^0 \subset X^1 \subset \cdots \subset X^n = X$$

for some $n \geqslant 0$ (called the dimension of X) where X^0 is a finite collection of points, and inductively each X^k (called the k-skeleton of X) is obtained from X^{k-1} by attaching finitely many k-cells.[†] The attaching maps of the cells can be *arbitrary* continuous maps, and are usually considered as part of the structure of the cell complex (see Hatcher pp. 5–8).

Remark A simplicial complex with oriented simplices is just a cell complex in which those simplices are viewed as cells whose attaching maps are embeddings that map each proper face onto another simplex. Somewhere in between simplicial and CW complexes are Hatcher's Δ-complexes, a.k.a. semisimplicial complexes, which we do not treat here.

In general, a cell complex X (just like a simplicial complex) is the disjoint union of the interiors of its cells, which are embedded open balls in X, but the boundaries of its cells are not necessarily embedded. They may have been partially or totally collapsed by the attaching process.

Definition A cellular :.

Can we make sense of the boundaries of the cells in a cell complex? Any $(k+1)$-cell e in a cell complex X has an attaching map $f_e\colon \partial e \to X^k$. Although f_e need not be an embedding, we can still view its image as an integer linear combination of k-cells in X:

For each k-cell c in X, the quotient space

$$X^k_c \;:=\; X^k/(X^k - c^\circ)$$

obtained by crushing the complement of c in X^k to a point, is naturally identified with $c/\partial c = S^k$ (via inverse stereographic projection). Noting that $\partial e = S^k$, the composition of the attaching map f_e with the quotient map $q_c\colon X^k \to X^k_c$

$$\partial e \xrightarrow{f_e} X^k \xrightarrow{q_c} X^k_c$$

is a map $f_{ec}\colon S^k \to S^k$ whose induced homomorphism $\mathbb{Z} \to \mathbb{Z}$ in any homology theory (identifying $H_k(S^k)$ with $\mathbb{Z}$) is given by multiplication by an integer $d_{ec} = \deg(f_{ec})$, called the Brouwer degree of f_{ec}. In geometric terms, d_{ec} records how many times f_e wraps ∂e around c. See below for more details. Now define the boundary of the $(k+1)$-cell e in the cell complex X to be the following integer linear combination of k-cells in X:

$$\partial e \;=\; \sum_{k\text{-cells } c \text{ in } X} d_{ec}\, c\,.$$

The fundamental notion of the degree of a map between spheres of the same dimension was introduced by L.E.J. Brouwer in 1912, who showed that homotopic maps have equal degrees, and in fact $\deg\colon \pi_k(S^k) \to \mathbb{Z}$ is a group isomorphism (by a theorem of Hopf in 1925). For smooth maps $f\colon S^k \to S^k$ (which represent all homotopy classes) there is a nice formula $\deg(f) = \sum_x \operatorname{sign}(df_x)$, summed over all $x \in f^{-1}(y)$ for any regular value y of f. So intuitively, one can think of $\deg(f)$ as the number of times f wraps the sphere around itself. Some other basic properties of deg include:

[†] Recall from general topology that the space $X \cup_f e$ obtained by attaching a cell e to a space X via a map $f\colon \partial e \to X$ is the quotient space $(X \sqcup e)/(x \in \partial e \sim f(x) \in X)$. It is an interesting exercise to prove that if $g\colon \partial B \to X$ is any map homotopic to f, then $X \cup_g e$ is homotopy equivalent to $X \cup_f e$.

- $\deg 1\!\!1 = 0$ and $\deg(fg) = \deg(f)\deg(g)$
- $\deg(f) = 0$ if f is not surjective
- $\deg(f) = -1$ if f is a reflection.

From the last property one can conclude that the degree of any fixed point free map $S^k \to S^k$ is $(-1)^{k+1}$, and as consequences 1) S^k has a nonzero tangent vector field iff k is odd, and 2) if k is even, then $\mathbb{Z}_2$ is the only nontrivial groups that acts on S^k. For details see Hatcher §2.2.

Topological Spaces

For an arbitrary space X, the role of an n-simplex or n-cell is played by a singular n-simplex, meaning a (continuous) map $\sigma\colon \Delta^n \to X$ from the standard n-simplex into X (this map need not be one-to-one, draw pictures). The boundary of σ is defined just as in the simplicial complex setting to be the alternating sum of its faces: $\partial\sigma = \sum_{i=0}^n \sigma_i$, where the ith face $\sigma_i : \Delta^{n-1} \to X$ is the singular $(n-1)$-simplex given by $\sigma_i(t_0,\ldots,t_{n-1}) = \sigma(t_0,\ldots,t_{i-1},0,t_i,\ldots,t_{n-1})$, i.e. $\sigma_i = \sigma\circ f_i$ where $f_i : \Delta^{n-1} \longrightarrow \Delta^n$ is the unique affine map sending e_j to e_j for $j<i$, and to e_{j+1} for $j \geqslant i$.†

The Associated Chain Complexes

Fix X, a simplicial complex, CW complex, or topological space. In the simplicial case, choose arbitrarily an orientation for each simplex (one way to do this is to arbitrarily order the vertices and then use this ordering to induce an orientation on each n-simplex for $n>0$). In the other two cases, the orientations of the cells or singular simplices are automatic since balls are naturally oriented, as are the standard simplices.

Let $C_n = C_n(X)$ be the free abelian group generated by the n-simplices with their chosen orientations (for X a simplicial complex) or the n-cells or singular n-simplices with their natural orientations (for X a cell complex or an arbitrary topological space). In the simplicial or cellular case, a superscript may be added to specify which chain groups are being defined, so either C_n^Δ or C_n^{CW}. The elements of C_n, called n-chains in X, are thus integer linear combinations of the oriented (singular) n-simplices or n-cells in X.

Now the boundary map ∂ on the generating set for $C_n(X)$ (of (singular) simplices or cells) extends linearly to each chain group, giving boundary homomorphisms $\partial_n : C_n \to C_{n-1}$ with $\partial_n\partial_{n+1} = 0$ for every n. Tis was verified in the simplicial case in HW#17, and the same proof applies when X is just a space.†

Thus, depending on the context, this yields the simplicial/cellular/singular chain complex of X:

$$(C,\partial) = (C(X,A),\partial) \;:\quad \cdots \longrightarrow C_{n+1} \xrightarrow{\partial_n} C_n \xrightarrow{\partial_{n-1}} C_{n-1} \longrightarrow \cdots$$

with superscripts Δ or CW when needed. Its homology $H(C(X),\partial)$, also denoted $H(X)$ (again with superscripts Δ or CW if needed) is called the simplicial/cellular/singular homology of X.

Example ① Compute the simplicial homology of the circle S^1 viewing it as the boundary of the standard 2-simplex $[e_0,e_1,e_2]$, with three vertices e_0, e_1, e_2 and three edges $[e_1,e_2]$, $[e_0,e_2]$, $[e_0,e_1]$. Since the boundary of any edge $[e_i,e_j]$ is $e_j - e_i$, the associated chain complex is

$$0 \longrightarrow \mathbb{Z}^3 \xrightarrow{\partial_1} \mathbb{Z}^3 \longrightarrow 0 \qquad \text{where} \qquad \partial_1 = \begin{pmatrix} 0 & -1 & -1 \\ -1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$

† Recall that the standard n-simplex is the convex hull in $\mathbb{R}^n$ of the origin $e_0 = 0$ and the tips of the standard basis vectors $e_1,\ldots,e_n$, i.e. $\Delta^n = [e_0,\ldots,e_n]$. By abuse of notation, Hatcher writes σ_i as $\sigma|[e_0,\cdots,\hat{e}_i,\cdots,e_n]$.

† Verifying $\partial^2 = 0$ is more difficult in the cellular case. We do not do this here, although it is implicitly verified later in showing that the theories yield isomorphic homology groups for any space in the overlap between the categories on which the theories are defined.

which yields $H_0(S^1) = \mathrm{coker}(\partial_1)$ and $H_1(S^1) = \ker(\partial_1)$, both isomorphic to $\mathbb{Z}$, and all the other homology groups are trivial (consistent with HW#8). Indeed one computes $\mathrm{tor}(\partial_1) = (1, 1)$ and so the calculation follows immediately from the basic formula $(*)$ on page 13.

The cellular homology of S^1, viewed a CW complex with one 0-cell and one 1-cell, leads to the chain complex

$$\cdots \to 0 \to \mathbb{Z} \xrightarrow{\partial_1} \mathbb{Z} \to 0$$

with $\partial_1 = 0$, so immediately we obtain the same groups as in the simplicial case. The singular homology is hopeless to calculate this way; but once we verify the axioms in that case, we can just use the sequence of a pair or Mayer-Vietoris to compute as before.

HW#18 Let K be the 1-skeleton of a 3-simplex (i.e. all its vertices and edges). Compute $H_1(K)$.

Example ② Compute the simplicial homology of S^2, viewing it as the boundary of the standard 3-simplex $[e_0, e_1, e_2, e_3]$. It has four vertices e_0, e_1, e_2, e_3, six edges $[e_i, e_j]$ for $0 \leqslant i < j \leqslant 3$, and four faces $[e_i, e_j, e_k]$ for $0 \leqslant i < j < k \leqslant 3$, and so (ordering the simplices lexicographically) the associated chain complex is

$$0 \longrightarrow \mathbb{Z}^4 \xrightarrow{\partial_2} \mathbb{Z}^6 \xrightarrow{\partial_1} \mathbb{Z}^4 \longrightarrow 0$$

where

$$\partial_1 = \begin{pmatrix} -1 & -1 & -1 & 0 & 0 & 0 \\ 1 & 0 & 0 & -1 & -1 & 0 \\ 0 & 1 & 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{pmatrix} \qquad \text{and} \qquad \partial_2 = \begin{pmatrix} 1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & -1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

To compute the homology, we find $\mathrm{tor}(\partial_1) = \mathrm{tor}(\partial_2) = (1, 1, 1)$, and it follows from $(*)$ that $H_n(S^2)$ is infinite cyclic for $n = 0, 2$ and trivial otherwise (again consistent with HW#8).

The cellular homology of S^2, viewed a CW complex with one 0-cell and one 2-cell, leads to the chain complex

$$\cdots \to 0 \to \mathbb{Z} \xrightarrow{\partial_2} 0 \xrightarrow{\partial_1} \mathbb{Z} \to 0$$

so immediately we obtain the same groups as in the simplicial case. In fact essentially the same argument recovers the homology of any sphere with virtually no work.

HW#19 Find a triangulation of the real projective plane $\mathbb{R}P^2$ with 10 triangles and use this to compute its homology.

Such calculations in the simplicial case quickly become unwieldly because of the complexity of triangulations of even the simplest spaces; the previous homework was designed to give you an appreciation of this fact. For example, any triangulation of the 2-torus – say with v vertices, e edges and f faces – requires at least 14 faces. Indeed, each face has 3 edges and each edge is shared by 2 faces, so $3f = 2e$. Thus the Euler characteristic $\chi = v - e + f = v - e/3 = v - f/2$. Since $\chi = 0$ for the torus, this shows that $e = 3v$ and $f = 2v$. But it is clear that $e \leqslant \binom{v}{2} = v(v-1)/2$, so $v(v-1) \geqslant 2e = 6v$, whence $v(v-7) \geqslant 0$. Thus $v \geqslant 7$ and so $f \geqslant 14$.

HW#20 Show that any triangulation of a closed orientable surface of genus 2 must contain at least 21 faces. Find a general lower bound for a closed surface of Euler characteristic χ.

HW#21 Compute the homology of any connected closed surface using cellular homology.

3. Uniqueness: cellular homology

It is a fact that any two homology theories defined on the category of (finite) CW-pairs will agree on that category, that is, will assign isomorphic homology groups to any given CW-pair. We prove this here for the absolute homology groups, leaving the relative case to the reader.

Fix such a homology theory H_* and a CW-complex X. It suffices to show that $H_n(X)$ can be computed by a scheme depending only on the cell structure of X: For each integer $n \geqslant 0$, set

$$C_n(X) \ := \ H_n(X^n, X^{n-1})$$

where X^n is the n-skeleton of X, and define the boundary map $\partial_n : C_n(X) \longrightarrow C_{n-1}(X)$ to be the connecting homomorphism in the sequence of the triple (X_n, X_{n-1}, X_{n-2}), i.e. ∂_n is the composition

$$H_{n+1}(X^{n+1}, X^n) \xrightarrow{\partial} H_n(X^n) \xrightarrow{j_*} H_n(X^n, X^{n-1}).$$

Then $\partial_n \partial_{n-1} = j_* \partial \, j_* \partial = j_* 0 \, \partial = 0$, and so we have a chain complex

$$(C(x), \partial) \ : \ \cdots \to C_{n+1}(X) \xrightarrow{\partial_{n+1}} C_n(X) \xrightarrow{\partial_n} C_{n-1}(X) \to \cdots$$

Theorem 3.1. *$H_n(X)$ is isomorphic to $H_n(C(X), \partial)$ for every n.*

Lemma 3.2. $H_k(X^n, X^{n-1}) = \begin{cases} \mathbb{Z}^{c_n} & \text{if } k = n, \text{ where } c_n \text{ is the number of } n\text{-cells in } X \\ 0 & \text{otherwise} \end{cases}$

Proof. $H_k(X^n, X^{n-1}) = \widetilde{H}_k(X^n/X^{n-1})$, since (X^n, X^{n-1}) is a good pair. But $X^n/X^{n-1} \cong W$, the union of c_n copies of S^n wedged together at a point p (with regular neighborhood N), and so

$$\begin{aligned} H_k(X^n, X^{n-1}) &= \widetilde{H}_k(W) = \widetilde{H}_k(W, p) \qquad \text{by the sequence of the pair } (W, p) \\ &= \widetilde{H}_k(W, N) \qquad \text{by the 5-lemma} \\ &= \widetilde{H}_k(W - \operatorname{int} N, \partial N) \qquad \text{by excision} \\ &= \oplus_{c_n} \widetilde{H}_k(B^n, \partial B^n) = \oplus_{c_n} \widetilde{H}_k(S^n) = \begin{cases} \mathbb{Z}^{c_n} & \text{if } k = n \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

□

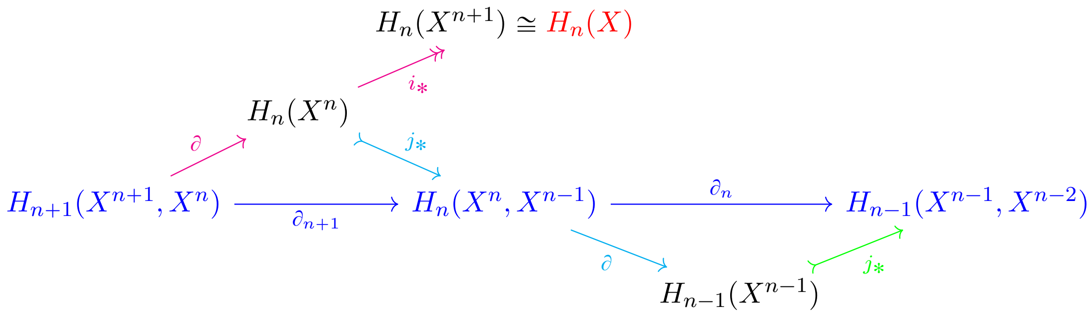
Corollary 3.3. $H_k(X^n) = \begin{cases} H_k(X) & \text{if } k < n \\ 0 & \text{if } k > n \end{cases}$*, and $H_n(X^n)$ has $H_n(X)$ as a quotient.*

Proof. Fix k, and consider the maps $i_n \colon H_k(X^{n-1}) \to H_k(X^n)$ induced by inclusions:

$$H_k(X^0) \to \cdots \to H_k(X^{k-1}) \xrightarrow{i_k} H_k(X^k) \xrightarrow{i_{k+1}} H_k(X^{k+1}) \to \cdots \to H_k(X)$$

Each i_n fits into an exact sequence $H_{k+1}(X^n, X^{n-1}) \to H_k(X^{n-1}) \xrightarrow{i_n} H_k(X^n) \to H_k(X^n, X^{n-1})$ whose first group vanishes if $n \neq k+1$ and whose last group vanishes if $n \neq k$, by the lemma. Thus i_n is an isomorphism if $n \neq k$ or $k+1$, and an epimorphism when $n = k+1$. Therefore, for $n < k$ we have $H_k(X^n) \cong H_k(X^0) = 0$ (since $k > 0$), for $n > k$ we have $H_k(X^n) \cong H_k(X)$, and $H_k(X^k)$ maps onto $H_k(X^{k+1}) \cong H_k(X)$. □

Proof. (of Theorem 3.1)



Since $H_n(X^{n+1}, X^n) = 0$ (by the Lemma), i_* is onto. Since $H_n(X^{n-1}) = 0 = H_n(X^{n-2})$ (by the Corollary) the j_* maps are one-to-one. Now by the Corollary

$$\begin{aligned} H_n(X) \;&\cong\; H_n(X^{n+1}) \;\cong\; H_n(X)/\mathrm{im}(\partial) \qquad \text{(by the first isomorphism theorem)} \\ &\cong\; \mathrm{im}(j_*)/\mathrm{im}(\partial_{n+1}) \;\cong\; \ker(\partial)/\mathrm{im}(\partial_{n+1}) \;\cong\; \ker(\partial_n)/\mathrm{im}(\partial_{n+1}) \;=\; H_n(C(X), \partial) \end{aligned}$$

□

4. Verifying the axioms

In this section, we establish the Eilenberg-Steenrod axioms for singular homology. The first three (functoriality of H and naturality of ∂) and the last (homology of a point) are straightforward, and left to the reader. Thus it remains to verify the homotopy, exactness and excision axioms.

Homotopy

Fix a homotopy $H : X \times I \to Y$ from $f : X \to Y$ to $g : X \to Y$. Then there is an associated chain homotopy $P\colon C(X) \to C(Y)$ from $f_\#$ to $g_\#$, defined as the linear extension of the map on singular n-simplices σ in X given by

$$P\sigma \;=\; \sigma^0 - \sigma^1 + - \cdots + (-1)^n\sigma^n$$

(note the superscripts, not subscripts). Here σ^i is the singular $(n+1)$-simplex in Y defined by

$$\sigma^i \;=\; H \circ (\sigma \times 1\!\!1) \circ t_i$$

where $t_i : \Delta^{n+1} \to \Delta^n \times I$ is the unique affine map given by

$$t_i(e_j) \;=\; \begin{cases} (e_j, 0) & \text{if } j \leqslant i \\ (e_{j-1}, 1) & \text{if } j > i \end{cases}$$

Thus $P\sigma$ is an alternating sum of singular $(n+1)$-simplices that are the images under $H \circ (\sigma \times 1\!\!1)$ of a decomposition of $\Delta^n \times I$. Draw picture.

One need only check that $\partial P + P\partial = g_\# - f_\#$, which we illustrate when $n = 2$: Label the bottom vertices $(e_i, 0)$ of $\Delta^2 \times I$ by a, b, c, and the correponding top vertices by A, B, C. We show that $\partial P = g_\# - f_\# - P\partial =$ top - bottom - sides:

$$\begin{aligned} \partial P \;&=\; \partial(aABC - abBC + abcC) \\ &=\; ABC - aBC + aAC - aAB - (bBC - aBC + abC - abB) + bcC - acC + abC - abc. \end{aligned}$$

The first and last terms are the top and bottom, and the rest (after cancelling four terms) are seen to be $-P\partial$.

Exactness

Snake

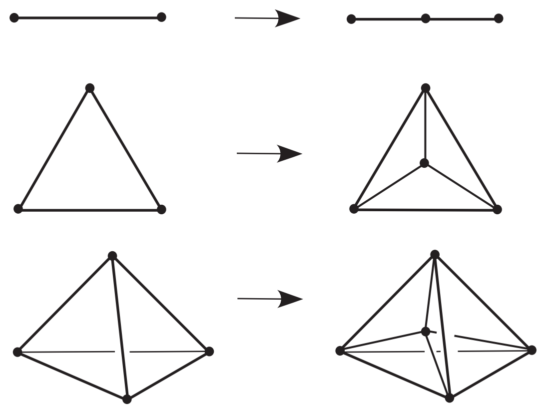
Excision

The rough idea is to show via "subdivision" how to represent elements of $H_n(X, A)$ by relative cycles whose simplicies are *small* relative to the set $U \subset A$ being excised, in the sense that they never stretch from $X - A$ to U, and then to discard the simplices that meet U.

Definition Let $\mathcal{U}$ be an open cover of X. A singular simplex $\sigma : \Delta^n \to X$ is $\mathcal{U}$-small if its image lies in some $U \in \mathcal{U}$, and a singular chain is $\mathcal{U}$-small if all of its constituent simplices are $\mathcal{U}$-small.

Subdivision Lemma. *If $\mathcal{U}$ is an open cover of X, then any class in $H_n(X, A)$ can be represented by a $\mathcal{U}$-small relative cycle.*

The proof uses barycentric subdivision, a systematic procedure for decomposing simplices into smaller ones:



To explain this precisely, define the barycenter (or center of mass) of the standard n-simplex $\Delta^n = [e_0, \ldots, e_n]$ to be the point

$$b_n \;=\; \sum_{i=0}^{n} \frac{1}{n+1}\, e_i \;\in\; \Delta^n\,,$$

and the barycenter of a singular simplex $\sigma : \Delta_n \to X$ to be the point $b_\sigma = \sigma(b_n)$ in X (so $b_{\mathbb{1}_{\Delta^n}} = b_n$).

Now consider a convex subset K of some Euclidean space (e.g. a simplex). For each n, let A_n denote the subgroup of the singular n chains in K generated by affine simplices $\sigma : \Delta^n \to K$, meaning $\sigma(\sum\; t_i e_i) = \sum t_i x_i$ where $x_i = \sigma(e_i)$. Note that if the x_i are not independent, then the image of σ will be a lower dimensional simplex. For notational convenience, we write $\langle x_1, \ldots, x_n\rangle$ for the map σ to distinguish it from its image $[x_0, \ldots, x_n]$.

Any point b in K defines a linear map $b\colon A_n \to A_{n+1}$, defined on affine simplices by

$$b\langle x_0, \ldots, x_n\rangle = \langle b, x_0, \ldots, x_n\rangle.$$

Note that $\partial b\sigma = \sigma - b\partial\sigma$, that is $\partial b + b\partial = \mathbb{1}$, and so the map b is in fact a chain homotopy from 0 to $\mathbb{1}$. This in turn gives rise to a chain map $S : A_* \circlearrowleft$ and a chain homotopy $T : A_* \circlearrowleft$ from S to $\mathbb{1}$, defined by $S = \mathbb{1}$ and $T = 0$ on 0-simplices, respectively, and inductively on any n-simplex σ (for $n > 0$) with barycenter b by

$$S\sigma = bS\partial\sigma \qquad \text{and} \qquad T\sigma = b(\mathbb{1} - T\partial)\sigma.$$

Indeed (supressing σ from the notation) we compute $\partial S = \partial bS\partial = S\partial - b\partial(S\partial) = S\partial - bS\partial^2 = S\partial$ and $\partial T = \partial b(\mathbb{1} - T\partial) = (\mathbb{1} - b\partial)(\mathbb{1} - T\partial) = \mathbb{1} - b\partial - T\partial + b\partial T\partial$. But by induction, the last term $b\partial T\partial = b(\mathbb{1} - S - T\partial) = b\partial - bS\partial - bT\partial^2 = b\partial - S$, and so

$$\partial T = \mathbb{1} - S - T\partial$$

as desired.

Now bootstrap ...

5. Relation between π_1 and H_1

Theorem 5.1. *If X is path connected, then $H_1(X)$ is the abelianization of $\pi_1(X,p)$.*[†]

Proof. The function $h : \pi_1(X,p) \to H_1(X)$ mapping $[\sigma]$ to $\overline{\sigma}$ is well defined, since any homotopy from σ to σ' yields a singular 2-simplex τ with $\partial\tau = p - \sigma' + \sigma$ (by collapsing $0 \times I$ in the domain of the homotopy to a point; draw the picture), so $\overline{\sigma} = \overline{\sigma}'$ (since $p = \partial$(trivial 2-simplex)).

Furthermore, h is a homomorphism: For any loops σ, σ' in X based at p, there is a singular 2-simplex τ with $\partial\tau = \sigma' - \sigma\cdot\sigma' + \sigma$ (draw picture), and so $\sigma + \sigma'$ is homologous to $\sigma\cdot\sigma'$.

Next we show that h is onto. For each $x \in X$, choose a path $\vec{x}$ from p to x, with reverse $\overleftarrow{x}$, where $\vec{p}$ is the constant path at p. Then any path $\alpha : I \to X$ determines a based loop $\vec{\alpha} = \vec{\alpha}_0 \cdot \alpha \cdot \overleftarrow{\alpha}_1$, and so the homology class of an arbitrary cycle $\sum \alpha_i$ is the image of the homotopy class of $\prod \vec{\alpha}_i$.

Finally, we show that $\ker(h) = \pi_1(X,p)'$. Since $H_1(X)$ is abelian, $\ker(h) \supset \pi_1(X,p)'$, so it suffices to show that any $[\gamma] \in \ker(h)$ is trivial modulo $\pi_1(X,p)'$. But working modulo $\pi_1(X,p)'$ allows one to rearrange the order of the factors in any decomposition of $[\gamma]$ as a product. The hypothesis $h([\gamma]) = 0$ implies that when γ, viewed as a 1-cycle, is the boundary of some 2-chain $\sum \tau_i$. That is,

$$\gamma \;=\; \partial(\textstyle\sum \tau_i) = \sum(\tau_{i0} - \tau_{i1} + \tau_{i2})$$

(also see NotesGluck/HermanSlides in my Papers/Notes/Books Folder)

6. The cohomology ring

6.1 Cohomology

6.2 Cup products

For any space X and any nonnegative integers p and q, we will define a linear map (i.e. group homomorphism) cup product

$$\cup : H^p(X) \otimes H^q(X) \;\longrightarrow\; H^{p+q}(X).$$

(or equivalently, a bilinear map $\cup : H^p(X) \times H^q(X) \;\longrightarrow\; H^{p+q}(X)$). The image of $a \otimes b$ is called the cup product of a and b, denoted $a \cup b$ or just ab. This product is associative, distributes over addition (this is what bilinearity means), and is "graded" commutative, i.e. $ab = (-1)^{pq}ba$. The cohomology class $1 \in H^0$, represented by the cocycle whose value on every 0-simplex is equal to 1, acts as the identity.

To define the cup product, we first treat the case of simplicial homology, where it is most easily defined on the cochain level $C^p(X) \otimes C^q(X) \to C^{p+q}(X)$. For technical reasons we initially order all the vertices of X, inducing orderings of all the simplices, and then define

$$(ab)[x_0, \ldots, x_{p+q}] \;=\; a([x_0, \ldots, x_k, 0, \ldots, 0])b([0, \ldots, 0, x_{n-k}, \ldots, x_n])$$

[†] The abelianization of a group G is the quotient group G/G', where G' is the commutator subgroup of G (also called its first derived subgroup) generated by all commutators $[x,y] := xyx^{-1}y^{-1}$ for $x, y \in G$. Note that G' is normal in G since $a[x,y]a^{-1} = [axa^{-1}, aya^{-1}]$. The natural projection $p : G \to G/G'$ is *initial* among all homomorphisms from G to abelian groups, i.e. characterized by the universal property that for any homomorphism $f : G \to A$ with A abelian, there is a unique homomorphism $g : G/G' \to A$ such that $f = g \circ p$. In particular $G' \subset \ker f$, so G/G' is the *largest* abelian quotient of G. Note that if G has a presentation in terms of generators and relation, then G' has the same presentation with the added relations that all pairs of generators commute.

The arguments of a and b on the right are called the front p-face and back q-face of the simplex $\sigma = [x_0, \ldots, x_{p+q}]$, also written ${}^p\sigma$ and σ^q, so with this notation the definition becomes $(ab)(\sigma) = a({}^p\sigma)b(\sigma^q)$. Intuitively, we compute ab on a simplex σ by "cupping" our hands around sigma, applying a with the left hand and b with the right, and then multiplying. Thus the name.

The coboundary maps $\delta : C^n(X) \to C^{n+1}(X)$ behave like the derivative (from calculus) with respect to the cup product. In particular, there is a product rule:

$$\delta(ab) = (\delta a)b + (-1)^p a(\delta b) \qquad (\text{for } a \in C^p)$$

This can be seen by a straightforward calculation, if a bit tedious, recalling that by definition $\delta a = a\partial$. The reader should at least verify this when $p = q = 1$.[†]

It follows that if both a and b are cocycles, then so is $a \cup b$, and if either one is a fortiori a coboundary, then so is $a \cup b$. Thus $\cup$ induces a product on cohomology, defined by $\overline{a}\overline{b} = \overline{ab}$ (i.e. $(a + B^p)(b + B^q) = ab + B^{p+q}$). The algebraic properties of $\cup$ follow easily from analogous properties at the cochain level, except commutativity (which only occurs at the cohomology level, and is harder to verify; see for example Theorem 3.14 in Hatcher).

Unfortunately, no such simple cochain level definition for the cup product in cellular homology is known. For singular homology, however, we just define the front p-face and back q-face of a singular simplex σ to be ${}^p\sigma = \sigma \circ f^p$ and $\sigma^q = \sigma \circ b^q$, where

$$f_k,\ b_k \colon \Delta^k \to \Delta^n$$

are the simplicial maps (i.e. unique "linear" extensions of maps f that send vertices to vertices, i.e. $f(\sum t_i e_i = \sum t_i f(e_i))$ that send the vertices of Δ^k (in order, i.e. $e_0, \ldots, e_k$) to the first/last k vertices of Δ^n. Then proceed as in the simplicial case.

How to compute cup products

Here are two ways:

1) (simplicial complexes) By hand, from the definition. Here's an example: $X = \mathbb{R}P^2$ with triangulation a hexagon with opposite sides identified, all oriented counterclockwise, and an internal concentric equilateral triangle with each of its vertices joined by three edges to the nearest three vertices of the hexagon. We work over $\mathbb{Z}_2$ to avoid orientations. Looking at the chain complex $\mathbb{Z}^{10} \to \mathbb{Z}^{15} \to \mathbb{Z}^6$, one can (fleshing out homework # 19) find generators for $H_n(\mathbb{R}P^2) = \mathbb{Z}_2$ for $n = 0, 1$ and 2 to be any vertex, the sum of the three edges on the boundary of the hexagon, and the sum μ of all the triangles. The "dual" generator of $H^1(\mathbb{R}P^2)$ is given by intersecting with a diameter x transverse to the triangulation, crossing through five triangles. For any choice of an ordering for the vertices, we find that x intersects both the front and back faces of an odd number of these five triangles, so evaluates to 1 on μ, so is nontrivial. Thus the cohomology ring is isomorphic to the "truncated" polynomial ring $\mathbb{Z}_2[x]/(x^3)$.

Maybe do another example: $X = T^2$, say triangulated as a square with an internal concentric square (plus diagonal), again with the vertices of the inner square joined with the three nearest vertices of the outer square.

These examples should be illustrated with pictures . . .

2) (smooth manifolds) Differential Topology: Intersection pairing (via Poincaré duality), or wedge products of forms (via the DeRham theorem).

There are probably other ways known to algebraic topology patricians.

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† Evaluating the left side on [0123] gives $\delta(ab)[0123] = (ab)(\partial[0123]) = (ab)([123] - [023] + [013] - [012]) = \mathbf{a[12]b[23]} - \mathbf{a[02]b[23]} + \mathbf{a[01]b[13]} - \mathbf{a[01]b[12]}$, and on the right, $((\delta a)b - a(\delta b))[0123] = a\partial[012]b[23] - a[01]b\partial[123] = (\mathbf{a[12]} - \mathbf{a[02]} + \mathbf{a[01]})\mathbf{b[23]} - \mathbf{a[01]}(\mathbf{b[23]} - \mathbf{b[13]} + \mathbf{b[12]})$ which equals the left side.

7. The basic theorems of algebraic topology

- **Universal Coefficient Theorems (UCT)** : relate homology and cohomology groups with arbitrary coefficients to integral homology
- **Kunneth Formulas** : compute homology and cohomology of product spaces
- **Hurewicz Theorem** : relate homology and homotopy groups
- **Whitehead Theorems** : relate homology equivalence, "weak" homotopy equivalence, and homotopy equivalence
- **Poincaré and Leftshetz Duality** : relate homology and cohomology of manifolds
- **Freudenthal Suspension Theorem** : (classical statement, 1937) for fixed k, the homotopy groups of spheres, $\pi_{n+k}S^n$, are independent of n for n sufficiently large (in particular, for $n \geqslant k+2$).

In this section we carefully state these theorems, and give some proofs and applications.

6.1 Universal Coefficient Theorems

For any space X, abelian group G, and integer n, there exist split short exact sequences

- (homology) $0 \to H_n(X) \otimes G \to H_n(X;G) \to H_{n-1}(X) * G \to 0$
- (cohomology) $0 \to \mathrm{Ext}(H_{n-1}(X), G) \to H^n(X;G) \to \mathrm{Hom}(H_n(X), G) \to 0$

The default coefficients are $\mathbb{Z}$, so $H_n(X) = H_n(X;\mathbb{Z})$. Note: $A * B$ is often written $\mathrm{Tor}(A, B)$.

The functors $\otimes$ and Hom, and their "derived" functors $*$ and Ext, all map $\mathbf{Ab} \times \mathbf{Ab} \to \mathbf{Ab}$, where $\mathbf{Ab}$ is the category of abelian groups. They distribute over $\oplus$ in both variables; $\otimes$ and $*$ are commutative, while Hom and Ext are not. When dealing with finitely generated abelian groups A, the operational definitions for $\otimes$ and $*$ are:

$$A \otimes \mathbb{Z} \cong A \ , \ \mathbb{Z}_p \otimes \mathbb{Z}_q \cong \mathbb{Z}_d \quad \text{and} \quad A * \mathbb{Z} = 0 \ , \ \mathbb{Z}_p * \mathbb{Z}_q \cong \mathbb{Z}_d$$

where $d = \gcd(p, q)$, and for Hom and Ext they are given by the following tables (where $\mathrm{Hom}(A, B)$ is the entry in the Hom table in row A and column B, etc.):

Hom	$\mathbb{Z}$	$\mathbb{Z}_q$
$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}_q$
$\mathbb{Z}_p$	0	$\mathbb{Z}_d$

Ext	$\mathbb{Z}$	$\mathbb{Z}_q$
$\mathbb{Z}$	0	0
$\mathbb{Z}_p$	$\mathbb{Z}_p$	$\mathbb{Z}_d$

Examples ① When $G = \mathbb{Z}$ and $H_*(X)$ is finitely generated, UCT for cohomology becomes

$$H^n(X) \;\cong\; \mathrm{Free}(H_n(X)) \oplus \mathrm{Tor}(H_{n-1}(X))$$

We provide a proof of the UCT in this case below.

② The integral homology $H_n(K)$ of the Klein bottle K is $\mathbb{Z}$ for $n = 0$, $\mathbb{Z} \oplus \mathbb{Z}_2$ for $n = 1$, and zero otherwise. Thus the integral cohomology $H^n(K)$ is $\mathbb{Z}$ for $n = 0$ or 1, $\mathbb{Z}_2$ for $n = 2$, and zero otherwise. For $\mathbb{Z}_q$ coefficients, we have $H^n(K;\mathbb{Z}_q) \cong \mathrm{Ext}(H_{n-1}(K), \mathbb{Z}_q) \oplus \mathrm{Hom}(H_n(K), \mathbb{Z}_q)$ so setting $d = \gcd(2, q)$ (which equals 2 or 0 according to whether q is even or odd),

$$H^0(K;\mathbb{Z}_q) = 0 \oplus \mathbb{Z}_q = \mathbb{Z}_q$$
$$H^1(K;\mathbb{Z}_q) = 0 \oplus (\mathbb{Z}_q \oplus \mathbb{Z}_d) = \mathbb{Z}_q \oplus \mathbb{Z}_d$$
$$H^2(K;\mathbb{Z}_q) = (0 \oplus \mathbb{Z}_d) \oplus 0 = \mathbb{Z}_d$$

and $H^n(K;\mathbb{Z}_q) = 0$ for $n > 3$.

Proof of the UCT for integral cohomology We begin with the definition of $\mathrm{Ext}(C,\mathbb{Z})$ for any abelian group C.[†] First observe that for any short exact sequence

$$(1) \qquad 0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$$

of abelian groups, the dual sequence

$$(1^*) \qquad 0 \leftarrow A^* \xleftarrow{f^*} B^* \xleftarrow{g^*} C^* \leftarrow 0$$

is exact at B^* (by the exactness of (1) at B and the universal property of quotient groups) and C^*, but not necessarily at A^*. For example the dual of $0 \to \mathbb{Z} \xrightarrow{\times 2} \mathbb{Z} \to \mathbb{Z}_2 \to 0$ is $0 \leftarrow \mathbb{Z} \xleftarrow{\times 2} \mathbb{Z} \leftarrow 0 \leftarrow 0$, which is not exact at the first $\mathbb{Z}$.

Now for any abelian group C, choose a sequence (1) in which both A and B are free; such a sequence exists since C is the quotient of a free group on any generating set. Then define:

$$\boxed{\mathrm{Ext}(C,\mathbb{Z}) \;=\; \mathrm{coker}(f^*)}$$

This group can be viewed as the homology of (1^*) at A^*, so measures the inexactness of (1^*).

To show $\mathrm{Ext}(C,\mathbb{Z})$ well defined, note (using a diagram chase) that for any other exact sequence

$$(2) \qquad 0 \to A' \xrightarrow{f'} B' \xrightarrow{g'} C \to 0$$

with A' and B' free, there is a chain map τ from (1) to (2) that is unique up to chain homotopy. Since chain maps induce homomorphisms on homology, it follows from the usual "universal mapping property" argument that τ induces an isomorphism between $\mathrm{coker}(f^*)$ and $\mathrm{coker}(f'^*)$.

Examples $\mathrm{Ext}(\mathbb{Z}_p,\mathbb{Z}) = \mathbb{Z}_p$ can be computed from the sequence $0 \to \mathbb{Z} \xrightarrow{\times p} \mathbb{Z} \to \mathbb{Z}_p$, while $\mathrm{Ext}(\mathbb{Z},\mathbb{Z}) = 0$ can be computed from $0 \to 0 \to \mathbb{Z} \to \mathbb{Z} \to 0$.

Now to prove the UCT for a given space X, consider the split short exact sequence

$$0 \to Z_* \hookrightarrow C_*(X) \xrightarrow{\partial} B_* \to 0$$

of chain complexes (where $Z_* = Z_*(X)$ and $B_* = B_*(X)$, with differentials o taken to be zero). This yields a dual split short exact sequence, that on the nth level looks like

$$0 \leftarrow Z_n^* \leftarrow C_n^*(X) \xleftarrow{\partial^*} B_n^* \leftarrow 0\,.$$

The associated long exact sequence is $\cdots \leftarrow B_n^* \xleftarrow{i_n^*} Z_n^* \leftarrow H^n(X) \leftarrow B_{n-1}^* \xleftarrow{i_{n-1}^*} Z_{n-1}^* \to \cdots$ where $i_k : B_k \hookrightarrow Z_k$ are inclusions. It follows that there is a short exact sequence

$$0 \longrightarrow \mathrm{coker}(i_{n-1}^*) \longrightarrow H^n(X) \longrightarrow \ker(i_n^*) \longrightarrow 0.$$

Now $\mathrm{coker}(i_{n-1}^*) = \mathrm{Ext}(H_{n-1}(X),\mathbb{Z})$ (from the SES $0 \to B_{n-1} \xrightarrow{i_{n-1}} Z_{n-1} \to H_{n-1} \to 0$) and $\ker(i_n^*) = (H_n(X))^*$, which yields the UCT.

[†] The definition of $\mathrm{Ext}(C,D)$ in general is obtained from this definition simply by replacing $\mathbb{Z}$ by D, and thus the dual group $X^* = \mathrm{Hom}(X,\mathbb{Z})$ by $\mathrm{Hom}(X,D)$, and the dual $f^* : Y^* \to X^*$ of any homomorphism $f\colon X \to Y$ by $f^* : \mathrm{Hom}(Y,D) \to \mathrm{Hom}(X,D)$ (defined by $f^*(\phi) = \phi \circ f$ as usual).

For calculations in the $D = \mathbb{Z}$ case, recall that if X and Y are free abelian of finite rank and f is represented with respect to some choice of bases by the matrix A, then f^* is represented with respect to the dual bases by A^T.

6.2 Kunneth Formulas

For any spaces X and Y and integers n, there exist split short exact sequences:

- (homology) $0 \longrightarrow (H_*X \otimes H_*Y)_n \longrightarrow H_n(X \times Y) \longrightarrow (H_*X * H_*Y)_{n-1} \longrightarrow 0$
- (cohomology) $0 \longrightarrow (H^*X \otimes H^*Y)_n \longrightarrow H^n(X \times Y) \longrightarrow (H^*X * H^*Y)_{n+1} \longrightarrow 0$

where by definition, $(H_*X \otimes H_*Y)_n = \bigoplus_{p+q=n}(H_pX \otimes H_qY)$ and similarly for $H_*X * H_*Y$, etc.

Example If K is the Klein bottle, then

$$\begin{aligned} H_0(K \times K) &= \mathbb{Z} \\ H_1(K \times K) &= \mathbb{Z}^2 \oplus \mathbb{Z}_2^2 \\ H_2(K \times K) &= \mathbb{Z}^2 \oplus \mathbb{Z}_2^2 \\ H_3(K \times K) &= \mathbb{Z}_2 \end{aligned}$$

and $H_n(K \times K) = 0$ for $n > 3$.

6.3 Hurewicz Theorem

Recall the higher homotopy groups

$$\pi_n(X,p) \;=\; [(S^n,s),(X,p)]$$

which are honest groups for $n > 0$, while $\pi_0(X,p)$ is just the "based" set of path components of X. If X is path-connected, then we suppress p from the notation since the isomorphism type of $\pi_n(X,p)$ is then independent of p. We say X is n-connected if $\pi_k(X,p)$ is trivial for all $k \leqslant n$. Thus 0-connected = path connected, 1-connected = path connected and with trivial fundamental group), etc.

Now define the Hurewicz homomorphisms

$$h_n : \pi_n(X) \longrightarrow H_n(X)$$

sending $[f]$ to $f_*(\omega)$, where ω is the "positive" generator of $H_n(S^n)$.

Hurewicz Theorem. *If X is path connected, then $h_1 : \pi_1(X) \to H_1(X)$ is the abelianization homomorphism*[†]*, and if X is $(n-1)$-connected for some $n > 1$, then $h_n : \pi_n(X) \to H_n(X)$ is an isomorphism.*

Exercise As an application, show that S^n is $(n-1)$-connected and that $\pi_n(S^n) \cong \mathbb{Z}$ for all $n > 0$. Also (using the homotopy sequence of a fibration) show that $\pi_2(\mathbb{C}P^n) \cong \mathbb{Z}$ for $n > 0$.

6.4 J.H.C. Whitehead Theorems

Definition A map $f : X \to Y$ is called an n-homotopy (resp. n-homology) equivalence if it induces isomorphisms on π_k (resp. H_k) for all $k < n$, and an epimorphism on π_n (rest. H_n). An ∞-homotopy equivalence is also called a weak homotopy equivalence.

Whitehead Theorem 1. *For any n (including ∞), a map is an n-homology equivalence if and only if it is an n-homotopy equivalence.*

Whitehead Theorem 2. *Any weak homotopy equivalence between* CW*-complexes is a homotopy equivalence.*

[†] i.e. it is onto with kernel equal to the commutator subgroup of $\pi_1(X)$

6.5 Duality Theorems for Manifolds

For now, a compact m-manifold M will be said to be orientable if and only if $H_m(M, \partial M) \cong \mathbb{Z}$.

Poincaré Duality. *If X is a closed, orientable m-manifold, then $H^k(X) \cong H_{m-k}(X)$ for all k.*

Remark First stated without proof by Poincaré in 1893. The analogous theorem holds with $\mathbb{Z}_2$-coefficients without the orientability hypothesis. Generalization (circa 1926):

Lefschetz Duality. *If X is a compact, orientable m-manifold, then $H^k(X) \cong H_{m-k}(X, \partial X)$ and $H^k(X, \partial X) \cong H_{m-k}(X)$ for all k.*

Remark More generally $H^k(X, Y) \cong H_{m-k}(X, Z)$ for all k, for any decomposition $\partial X = Y \cup Z$, where Y and Z are disjoint unions of components of ∂X.

Exercises Use the theorems above to show ① Any 1-connected closed 3-manifold Y is homotopy equivalent to the 3-sphere. (It is now known via Perelman's work that Y is in fact homeomorphic to the 3-sphere.) ② The boundary of any compact, orientable 3-manifold Y for which $H_1(Y)$ is finite is a union of 2-spheres. ③ Describe the homology of a simply-connected 4-manifold. ④ How much of the homology of a closed orientable m-manifold M is needed (working from the bottom up) to determine all of its homology?

Sketch of a geometric proof of Poincaré duality For simplicity, assume M has a *combinatorial* triangulation T,† and let (C^*, δ) denote its associated cochain complex (that can be used to compute the cohomology of M). Note that C^* has a natural basis consisting of the duals σ^* of the simplices σ in the triangulation; σ^* assigns 1 to σ and 0 to all other simplices.

The idea is to construct a cell decomposition D dual to T, whose cellular chain complex (D, ∂) is isomorphic to (C^*, δ). In particular we will construct D and a natural isomorphism $\phi\colon C^* \to D$ of chain complexes (meaning $\partial\phi = \phi\delta$) mapping each dual k-simplex to an $(m-k)$-cell in D. It follows that $H^k(M) = H^k(C^*, \delta) \cong H_{m-k}(D, \partial) = H_{m-k}(M)$, which will prove Poincaré duality. So what is this CW structure D on M dual to T, and what is the chain isomorphism ϕ?

Given simplices σ and τ in T, we write $\sigma < \tau$ if σ is a face of τ, and then call τ a carrier of σ. The dual bit $\widehat{\sigma}_\tau$ of σ in τ is the convex hull of the barycenters of all carriers of σ in τ, i.e. all the simplices ω with $\sigma < \omega < \tau$. Draw the pictures when $m \leqslant 3$. The dual cell $\widehat{\sigma}$ of any simplex σ in T is the union of all its bits in all its carriers,

$$\widehat{\sigma} \;=\; \cup_{\tau > \sigma} \widehat{\sigma}_\omega.$$

Again draw some pictures for $m \leqslant 3$.

It can be proved that if σ is a k-cell, then $\widehat{\sigma}$ is an $(m-k)$-cell that intersects σ transversely in one point (this uses the fact that T is combinatorial, otherwise $\widehat{\sigma}$ need not even be a manifold) and that these dual cells fit together to give a CW structure on M (a particularly nice one since the closed cells are all embedded). This is the dual cell decomposition D.

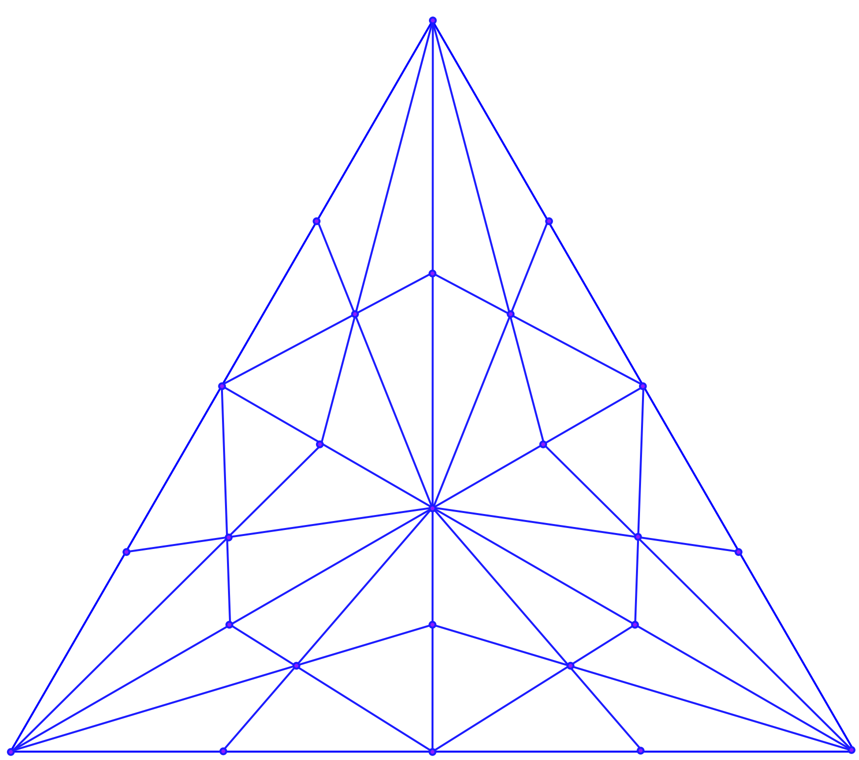
The key observation is that the boundary in (D, ∂) of the dual cell of a k-simplex σ is the sum (maybe with signs if we consider orientations?) of the dual cells of all its $(k+1)$-dimensional carriers

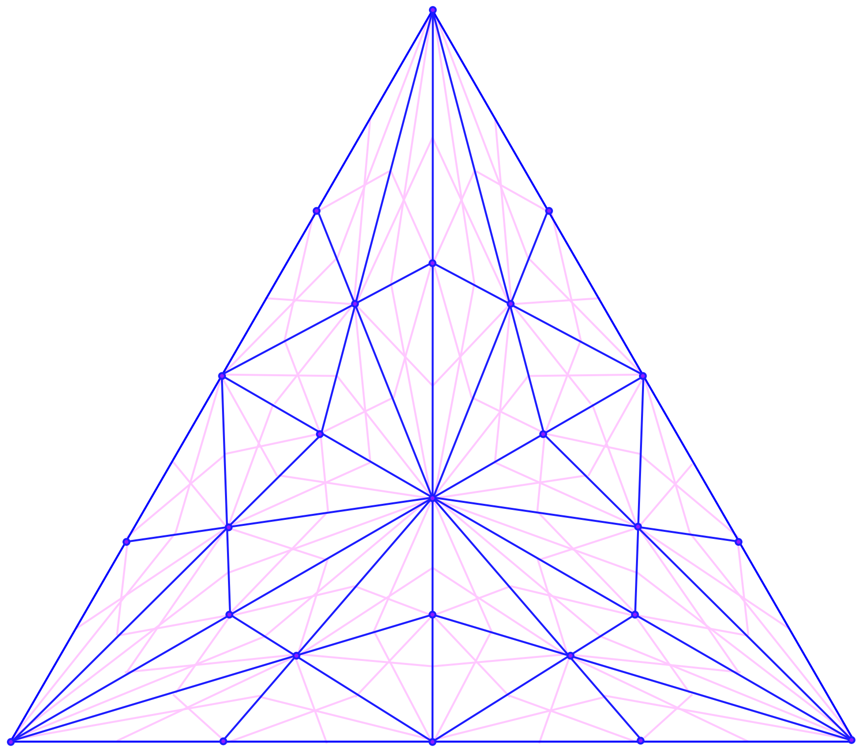
$$\partial\widehat{\sigma} \;=\; \sum_{\tau^{k+1} > \sigma} \widehat{\tau}$$

† The "combinatorial" condition means the star of every vertex in T is a triangulation of the m-ball. Not all triangulable manifolds have combinatorial triangulations (Edwards I think) but all smooth manifolds do. Perhaps even more surprisingly, not all manifolds are triangulable (Kirby/Siebenmann/Manolescu I think).

Draw some pictures when $m = 2$ to show that this is plausible, although the proof is technical. Now just define $\phi(\sigma^*) = \widehat{\sigma}$. Then $\partial\phi(\sigma^*) = \partial\widehat{\sigma} = \sum \hat{\tau} = \phi(\sum \tau^*) = \phi\delta(\sigma^*)$, as desired.

Some relevant pictures: T and its first barycentric subdivision T'





7.6 Freudenthal Suspension Theorem (for the case of spheres)

For a sketch of a quick differential topology proof, using the Pontrjagin-Thom construction, see Francisco Lin's posting at mathoverflow.net/questions/56435 :

At the top: The Freudenthal suspension theorem states in particular that the map

$$\pi_{n+k}(S^n) \to \pi_{n+k+1}(S^{n+1})$$

is an isomorphism for $n \geqslant k+2$. My questions is: What is the intuition behind the proof of the Freudenthal suspension theorem?

Francesco's answer: Maybe this differential topologic way of thinking the Freudenthal suspension is much more intuitive. By Pontrjagin's contruction you can identify $\pi_{n+k}(S^n)$ with equivalence classes of framed submanifolds (N, ν) of S^{n+k}. The image of this class under the Freudenthal suspension homomorphism is just the framed submanifold $(N, \tilde{\nu})$, where we identify S^{n+k} with the equator of S^{n+k+1} and the frame $\tilde{\nu}$ is obtained by ν just "adding" to ν the canonical normal frame of S^{n+k} inside S^{n+k+1}. The fact that the map is an isomorphism for $n > k+1$ can now be achieved by general position arguments.

Maybe flesh this out ... ?