# PDE Lecture Notes

Based in part on Rhonda Hughes' Lecture Notes
Text: Partial Differential Equations by Walter Strauss

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## 0 Introduction

Partial differential equations (PDEs) arise as models of physical, biological and economic phenomena (e.g. Maxwell's equations for the electromagnetic field, Schrödinger's equation in quantum theory, Black-Scholes equation for pricing derivatives) and are also fundamental tools in many areas of pure and applied mathematics. The use of PDEs is a two-fold process: One must first formulate a PDE for a given problem (i.e. construct the mathematical model), and then figure out how to solve it. The former is an art; we will focus on the latter.

Some history: In the 18th century, basic problems in continuum mechanics (vibrating strings, fluid dynamics) were formulated by Euler, Bernoulli and d'Alembert as PDEs, but it was not until Fourier's treatise in 1807 that a systematic means of solving these problems was proposed. Fourier's paper, although initially rejected, was awarded a prize after its resubmission in 1812. His definitive book on the subject was published in 1822, giving uniform solutions to many of the major PDEs of the time including the wave and heat equations, and Laplace's equation.

These three equations (with auxiliary initial or boundary conditions) will be the focus of this course. Along the way, we will develop the basics of Fourier analysis, a subject which occupies a central role in applied mathematics as well as many areas of pure mathematics, including representation theory, number theory, geometry and topology. Fourier analysis has also stimulated the development of many new fields, e.g. wavelet analysis.

### What is a PDE?

A PDE is an equation containing partial derivatives of an unknown multivariable function (instead of single-variable function as for ODEs). It relates the independent variables $t, x, y, \ldots$ (where $t$ always denotes time, while $x, y, \ldots$ typically denote "spatial" variables), the dependent variable $u$, and the partial derivatives of $u$ [†] and so can be written

$$F(x, t, u, u_x, u_t) = 0 \qquad \text{or} \qquad F(x, t, u_x, u_t, u_{xx}, u_{xt}, u_{tt}) = 0 \qquad \text{etc.}$$

for suitable $F$. A solution to the PDE is a function $u(x, y, \ldots)$ that satisfies this equation, at least in some region of the independent variable space.

[†] Often denote partial derivatives by subscripts: $u_x = \frac{\partial u}{\partial x}$, $u_{xy} = \frac{\partial^2 u}{\partial y \partial x}, \ldots$ (Recall that if $u$ is $C^2$, i.e. twice continuously differentiable, then all mixed partials are equal: $u_{xy} = u_{yx}$, etc.) Also write $\partial_x$ as shorthand for the "differential operator" $\partial/\partial x$, etc., so for example $u_x = \partial_x u$ and $u_{xy} = \partial_{xy} u = \partial_y \partial_x u$.

**Examples** † (all with two independent variables $x$ and $t$, or $x$ and $y$)

1. (transport) $u_t + cu_x = 0$ (where $c$ is a constant)
2. (wave equation) $u_{tt} = c^2 u_{xx}$
3. (diffusion/heat equation) $u_t = ku_{xx}$ (where $k > 0$ is constant)
4. (Laplace's equation) $u_{xx} + u_{yy} = 0$
5. (Poisson's equation) $u_{xx} + u_{yy} = f(x, y)$ (for any function $f$)
6. (Schrödinger's equation) $u_t = i\hbar u_{xx} + f(x)u$ (where $i = \sqrt{-1}$)
7. (Black-Scholes equation) $u_t + x^2 u_{xx} + xu_x - u = 0$
8. (Klein-Gordon equation) $u_{tt} - c^2 u_{xx} + m^2 u + gu^3 = 0$
9. (Korteweg-deVries equation) $u_t + u_{xxx} + 6uu_x = 0$

**Terminology**

**Order** The order of a PDE is the order of the highest partial derivative that occurs. For example equation 1 is first order, 2–8 are second order, and 9 is third order.

**Linearity** A PDE is called homogeneous linear if, when written in the form $\mathcal{L}(u) = 0$ (where $\mathcal{L}$ is a differential operator, e.g. $\mathcal{L} = \partial_t - k\partial_{xx}$ in the heat equation), the operator $\mathcal{L}$ is linear, meaning

$$\mathcal{L}(u + v) = \mathcal{L}(u) + \mathcal{L}(v) \qquad \text{and} \qquad \mathcal{L}(cu) = c\,\mathcal{L}(u)$$

for any functions $u$ and $v$, and constant $c$. What this really means is that $\mathcal{L}(u)$ is a linear combination of $u$ and its partial derivatives with coefficients that are constants or functions of the independent variables. A PDE of the form $\mathcal{L}(u) = f(x, t, \dots)$ with $\mathcal{L}$ linear and $f$ not identically zero is called an inhomogeneous linear PDE. For example, equations 1–7 are linear (all homogeneous except Poisson's equation) while 8 and 9 are not (because of the last term in each case: $(cu)^3 \neq cu^3$ and $(cu)(cu)_x \neq cuu_x$).

†Equations 2–6 and 8 can be framed in higher dimensions, replacing $u_{xx}$ $(+\,u_{yy})$ by the Laplacian of $u$

$$\Delta u := u_{xx} + u_{yy} + \cdots \qquad \text{(summed over all the spatial variables } x, y, \dots)$$

and $f$ by a (suitable) function of all the spacial variables.

Recall from vector analysis that the Laplacian operator $\Delta := \partial_{xx} + \partial_{yy} + \cdots$ can also be written as the "square" $\nabla^2 = \nabla \cdot \nabla$ of the familiar "del operator" $\nabla = (\partial_x, \partial_y, \cdots)$, and

$$\text{grad}(u) = \nabla u \qquad \text{div}(\mathbf{v}) = \nabla \cdot \mathbf{v} \qquad \text{curl}(\mathbf{v}) = \nabla \times \mathbf{v}$$

for any scalar field $u$ and vector field $\mathbf{v}$ in $\mathbb{R}^3$. This notation is generally used to describe PDEs in higher dimensions, including Maxwell's equations for the electric and magnetic fields $\mathbf{E}$ and $\mathbf{B}$ (in a vacuum):

$$\mathbf{E}_t = c\,\nabla \times \mathbf{B} \quad \text{and} \quad \mathbf{B}_t = -c\,\nabla \times \mathbf{E} \quad \text{and} \quad \nabla \cdot \mathbf{E} = \nabla \cdot \mathbf{B} = 0$$

where $c$ is the speed of light. This is a system of twelve PDEs involving six unknown functions (the components of $\mathbf{E}$ and $\mathbf{B}$) of four independent variables ($x, y, z$ and $t$).

Why is linearity important?

Principle of superposition The solutions of a homogeneous linear PDE form a vector space, ie. if $u_1, \dots, u_n$ are solutions and $c_1, \dots, c_n$ are scalars, then $\sum c_i u_i$ is also a solution.

It will be seen in the next section that solving a first order linear PDE reduces to the solution of some associated first order (but not necessarily linear) ODEs, and in many cases of interest this is easily accomplished.[†]

Our focus in this course will be on second order linear PDEs in two independent variables

$$Au_{xx} + 2Bu_{xy} + Cu_{yy} + Du_x + Eu_y + Fu = G$$

where A–G are constants or functions of $x$ and $y$. (The reason for the 2 will be seen later.) Such equations, which include many of the important PDEs in applications, are classified into three types according to the sign of $\Delta = AC - B^2$:

1. elliptic if $\Delta > 0$
2. parabolic if $\Delta = 0$
3. hyperbolic if $\Delta < 0$

The prototypes are Laplace's equation, the heat equation, and the wave equation, resp. Indeed any second order linear PDE with constant coefficients can be transformed into one of these by a suitable change of variables (see below). If the coefficients are functions, then of course the type of the PDE may vary in different regions of the independent variable space.

The solutions for these three types of PDEs have very different characters. For example, it will be seen below that for the heat equation $u_t = ku_{xx}$, a local variation in the initial conditions will be felt instantly at any distance, in contrast to the wave equation $u_{tt} = c^2 u_{xx}$ in which the propagation speed for such a variation is finite (indeed equal to $c$).

## Linear change of variables

A change of variables $x, y \rightsquigarrow r, s$ can often help solve a PDE. This change is linear if it is of the form $r = ax + by$ and $s = cx + dy$ with $ad - bc \neq 0$ (the last condition $\Longrightarrow$ the change is reversible). Now the chain rule gives $u_x = u_r r_x + u_s s_x = au_r + bu_s$ and $u_y = u_r r_y + u_s s_y = bu_r + du_s$. In other words

$$\partial_x = a\partial_r + c\partial_s \qquad \text{and} \qquad \partial_y = b\partial_r + d\partial_s$$

Solving for $\partial_r, \partial_s$ in terms of $\partial_x, \partial_y$ (for example using matrices) gives $\partial_r = m(d\partial_x - c\partial_y)$ and $\partial_s = m(-b\partial_x + a\partial_y)$ where $m = 1/(ad - bc)$. This shows that (after renaming the variables)

$$\text{substituting } \begin{cases} s = bx - ay \\ r = \text{anything} \end{cases} \quad \text{converts} \quad a\partial_x + b\partial_y \text{ into a multiple of } \partial_r \qquad (*)$$

Remember this as $s =$ inner $-$ outer (or a multiple); we'll call it the io (or oi) substitution. The usefulness of this substitution is illustrated in some of the examples below.

[†]It will be assumed that the reader is familiar with a few basic kinds of ODEs, including first order separable equations $y' = f(x)g(y)$ (solved by separating variables $dy/g(y) = f(x)dx$ and then integrating; e.g. $y' = ay$ has solution $y = ce^{ay}$) and first order linear equations $y' + p(x)y = q(x)$ (solved by multiplying through by the "integrating factor" $f(x) = \exp(\int p(x)dx)$, converting the equation into $(f(x)y)' = f(x)q(x)$, with solution $y = f(x)^{-1} \int f(x)q(x)dx$). Also the second order equation $y'' + c^2 y = 0$ (with solution $y = a\cos(cx) + b\sin(cx)$).

**Solutions to some simple PDEs** : Find all $u(x, y)$ satisfying

① $u_x = 0$. If this were an ODE, then one integration would give $u = c$ for constant $c$. Since this is a PDE (depending also on $y$) this constant can be any function $f(y)$ of $y$. So the general solution is

$$u(x, y) = f(y)$$

where $f$ is an arbitrary function of one variable. (We implicitly assume that $f$ is differentiable enough, once in this case, to be plugged into the PDE.)

② $2u_x + 3u_y = 0$. The change of variables $r = 2x + 3y$ and $s = 3x - 2y$ (the io-substitution) converts this equation into $u_r = 0$ (since $u_x = 2u_r + 3u_s$ and $u_y = 3u_r - 2u_s$ by the chain rule, so $2u_x + 3u_y = 13u_r = 0 \Longrightarrow u_r = 0$). This has solution $u = f(s)$, i.e.

$$u(x, y) = f(3x - 2y)$$

for arbitrary $f$ (verify this).

③ $u_{xx} = 0$. Integrating wrt $x$ gives $u_x = f(y)$, and then again wrt $x$ gives

$$u(x, y) = f(y)x + g(y)$$

for arbitrary $f$ and $g$.

④ $u_{xx} + u = 0$. The corresponding ODE has solution $u = a \cos x + b \sin x$ where $a$ and $b$ are constants, so the general solution to the PDE is

$$u(x, y) = f(y) \cos x + g(y) \sin x.$$

for arbitrary $f$ and $g$.

⑤ $u_{xy} = 0$. Integrating wrt $y$ gives $u_x = h(x)$, and then wrt $x$ gives $f(x) + g(y)$ where $f' = h$. So the general solution is

$$u(x, y) = f(x) + g(y)$$

for arbitrary $f$ and $g$. As we will see in §3 below, the unbounded wave equation reduces to this equation by a simple change of variables.

⑥ $u_{xx} - u_{xy} - 2u_{yy} = 0$. The change of variables $r = x - y$ and $s = 2x + y$ (the io-substitution applied twice since the operator $\partial_{xx} - 2\partial_{xy} - \partial_{yy}$ factors as $(\partial_x + \partial_y)(\partial_x - 2\partial_y)$; might as well call it the oioi-substitution!) converts this equation into $u_{rs} = 0$. This has solution $u = f(r) + g(s)$, i.e.

$$u(x, y) = f(x - y) + g(2x + y)$$

for arbitrary $f$ and $g$. The reader should verify that this is indeed a solution.

Moral Solutions of PDEs involve arbitrary functions (instead of constants as for ODEs).

# 1 First order linear equations

We only treat the case of two independent variables

$$a(x,y)u_x + b(x,y)u_y + p(x,y)u = q(x,y)$$

although similar methods work in general.

First assume $p = q = 0$

Case 1 Constant coefficients $\quad au_x + bu_y = 0 \quad$ (with $a, b \neq 0$)

Two approaches:

**Coordinate Method** (a.k.a. the oi-substitution)

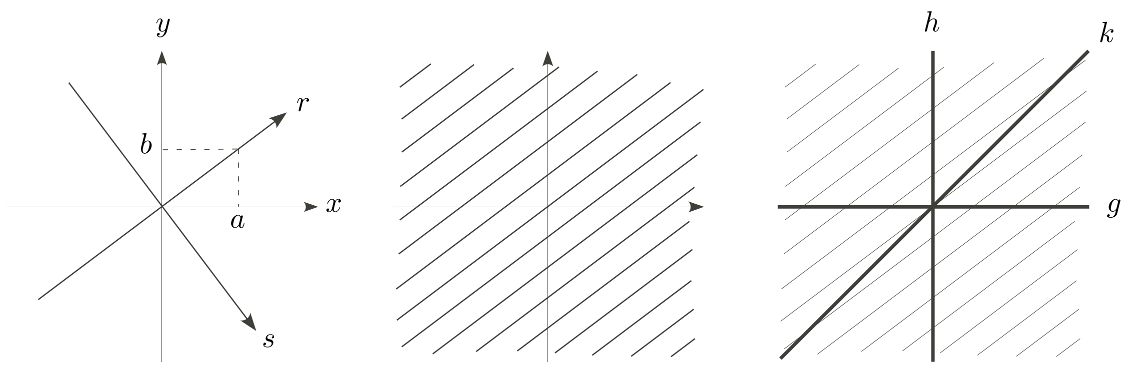
The change of variables $r = ax+by$, $s = bx-ay$ converts the PDE into $u_r = 0$ (to review, this is because $u_x = au_r+bu_s$ and $u_y = bu_r-au_s$ by the chain rule, so $au_x+bu_y = (a^2+b^2)u_r$) which has solution $u = f(s)$, i.e.

$$u(x,y) = f(bx - ay)$$

for arbitrary $f$. (See Figure 1(a))

**Geometric Method** (a.k.a. method of characteristics)

The expression $au_x+bu_y$ is the directional derivative $\nabla_{(a,b)}u$ of $u$ in the direction $(a,b)$, and so the PDE says $u$ is constant along all lines parallel to $(a,b)$, the so-called characteristic lines of the equation. These lines (parallels to the $r$-axis; Figure 1(b)) have equations $bx - ay = c$ (constant) and so the solution is of the form $u = f(c) = f(bx - ay)$ as we found above.




(a) new coordinates (b) characteristic lines (c) auxiliary conditions

Figure 1: Constant coefficients

The PDE may have an auxiliary condition such as $u(x,0) = g(x)$, specifying $u$ on the $x$-axis. This would mean $f(bx) = g(x)$, i.e. $f(w) = g(w/b)$, giving a unique solution $u(x,y) = g(x - ay/b)$ to the equation. Similarly the condition $u(0,y) = h(y)$ specifies $u$ on the $y$-axis and yields the solution $u(x,y) = h(y - bx/a)$, and $u(x,x) = k(x)$ (assuming $a \neq b$) specifies $u$ on the line $y = x$ and yields $u(x,y) = k((bx - ay)/(b - a))$. (See Figure 1(c))

**Example** Find $u(x,t)$ satisfying the initial value problem (IVP)

$$2u_t + 3u_x = 0 \qquad \text{and} \qquad u(x,0) = \sin x.$$

Without the initial condition we get (by either method) $u(x,t) = f(2x-3t)$. Thus $u(x,0) = f(2x) = \sin x \Longrightarrow f(w) = \sin(w/2) \Longrightarrow$ the unique solution is $u(x,t) = \sin(x - 3t/2)$.

CASE 2 Variable coefficients $a(x,y)u_x + b(x,y)u_y = 0$

The coordinate method would require a nonlinear change of variables, so we use the geometric method instead.

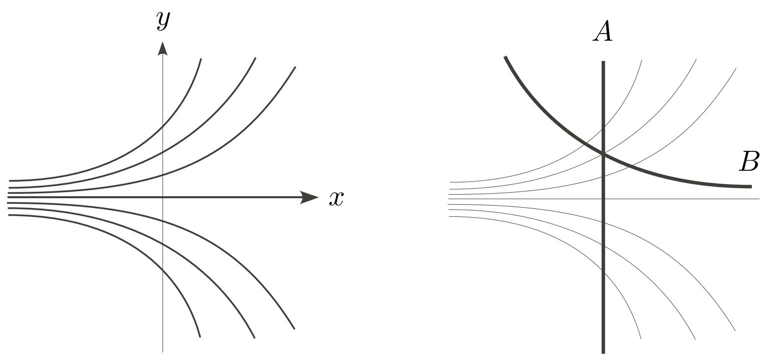
For example consider the equation

$$u_x + yu_y = 0 \qquad \text{or equivalently} \qquad \nabla_{(1,y)}u = 0$$

What curves have $(1,y)$ as tangent vectors? Curves with slope $y$, i.e. curves satisfying

$$dy/dx = y/1$$

which are $y = ce^x$. These are called the characteristic curves of the PDE (Figure 2(a)).




(a) characteristic curves (b) auxiliary conditions

Figure 2: Variable coefficients

Now $u(x,y)$ is constant on each curve, since by construction its derivative in the direction of its tangents vanishes,† so is given by its value at any chosen point on the curve. For example if we use the point of intersection with the $y$-axis then

$$u(x, ce^x) = u(0, ce^0) = u(0,c)$$

which can be an arbitrary function $f(c)$. But the value of $c$ corresponding to any point $(x,y)$ is $c = ye^{-x}$, so the general solution is

$$u(x,y) = f(ye^{-x}).$$

for arbitrary $f$.

†Can check this using the chain rule: $\frac{d}{dx}u(x, ce^x) = u_x + ce^x u_y = u_x + yu_y = 0$.

This method works in general, reducing the solution of $a(x,y)u_x + b(x,y)u_y = 0$ to the ordinary differential equation $dy/dx = a(x,y)/b(x,y)$. If the ODE can be solved, say in the form $h(x,y) = c$, then the general solution to the PDE is $u(x,t) = f(h(x,y))$ where $f$ can be an arbitrary function. This is because $u$ is constant along the solution curves $h(x,y) = c$ of the ODE. These are called the characteristic curves of the PDE.

Now any auxiliary condition, which typically specifies $u$ along some line or curve $A$ that intersects each characteristic at most once, will determine the solution uniquely in the region swept out by those characteristics that do in fact intersect $A$ (but not elsewhere). Simply impose the condition to determine $f$.

**Example** Solve $u_x + yu_y = 0$ (which is the PDE considered above) with each of the following auxiliary conditions: $(A)$ $u(0,y) = \sin y$ and $(B)$ $u(x, e^{-x}) = 2x$ (see Figure 2(b)) and discuss the uniqueness of your solution.

Solution Recall that the general solution is of the form $u(x,y) = f(ye^{-x})$. Thus for $(A)$ we have $u(0,y) = f(ye^0) = f(y) = \sin y$, and so

$$u(x,y) = \sin(ye^{-x})$$

is the unique solution in the whole plane (since every characteristic intersects the $y$-axis). For $(B)$ we have $u(x, e^{-x}) = f(e^{-x}e^{-x}) = f(e^{-2x}) = 2x \Longrightarrow f(w) = 2(-\frac{1}{2}\ln w) = \ln(1/w)$, and so

$$u(x,y) = \ln(e^x/y)$$

is the unique solution in the upper-half plane (the union of the characteristics that intersect the curve $y = e^{-x}$). Elsewhere $u$ can still be an arbitrary function of $ye^{-x}$.

Now assume $p$ and $q$ are arbitrary

We will only treat the case when $a$ and $b$ are constant: $au_x + bu_y + p(x,y)u = q(x,y)$.[†] Then the coordinate method (changing variables to $r = ax + by$ and $s = bx - ay$) still works, converting the equation into a linear first order ODE in the variable $r$ (holding $s$ constant)

$$(a^2 + b^2)u_r + P(r,s)u = Q(r,s)$$

where $P$ and $Q$ are obtained from $p$ and $q$ by substituting $x = (ar + bs)/(a^2 + b^2)$ and $y = (br - as)/(a^2 + b^2)$ (these formulas come from inverting the change of variables) and the constant of integration is replaced by a function of $s$ as usual. The solution of this ODE will be a function of $r$ and $s$, and substituting back in terms of $x$ and $y$ will then solve the PDE.

**Example** Find the unique solution $u(x,t)$ to the initial value problem

$$3u_t + 4u_x - 25u = 0 \qquad \text{and} \qquad u(x,0) = \sin(3x).$$

Solution Substitute $r = 4x+3t$ and $s = 3x-4t$ to get $u_r - u = 0$, with solution $u = f(s)e^r$, so $u(x,t) = f(3x-4t)e^{4x+3t}$ is the general solution of the PDE. The initial condition gives $f(3x)e^{4x} = \sin(3x) \Longrightarrow f(w) = e^{-4w/3}\sin w$ and so the unique solution to the IVP is

$$u(x,t) = e^{4x+3t-4(3x-4t)/3}\sin(3x-4t) = e^{25t/3}\sin(3x-4t)$$

as is easily verified.

[†] If $a$ and $b$ are variable, then the geometric method gives the characteristic curves as solutions to the ODE $dy/dx = b(x,y)/a(x,y)$ as before. To solve the PDE, one must parametrize each characteristic $C$ by some parameter $r$, and then solve the ODE $du/dr + p(r)u = q(r)$ along C.

**Application:** Simple transport

Consider a pollutant suspended in a fluid flowing at a constant speed $c$ in a horizontal pipe and let $u(x,t)$ be the concentration (mass/unit length) of the pollutant at position $x$ and time $t$. Assume that the pollutant is simply carried along (transported) with the fluid as it moves, without "diffusing". We claim that $u(x,t)$ obeys the PDE

$$\boxed{u_t + cu_x = 0}$$

To see this, note that during any time interval $[t, t+h]$ the fluid at position $x$ will move to position $x + ch$, and so the corresponding concentrations will be equal

$$u(x + ch, t + h) = u(x,t).$$

Now differentiating with respect to $h$ (using the chain rule on the left hand side) and putting $h = 0$ gives $cu_x + u_t = 0$ as claimed.

Now the solution (via the oi-substitution) is

$$u(x,t) = f(x - ct)$$

where $f(x)$ is the initial condition $u(x,0)$. Geometrically this is a wave of shape $u = f(x)$ traveling to the right at speed $c$, as could have been predicted without solving the PDE!

This can be illustrated by drawing a few snapshots of $u(x,t)$ for increasing values of $t$, as in Figure 3

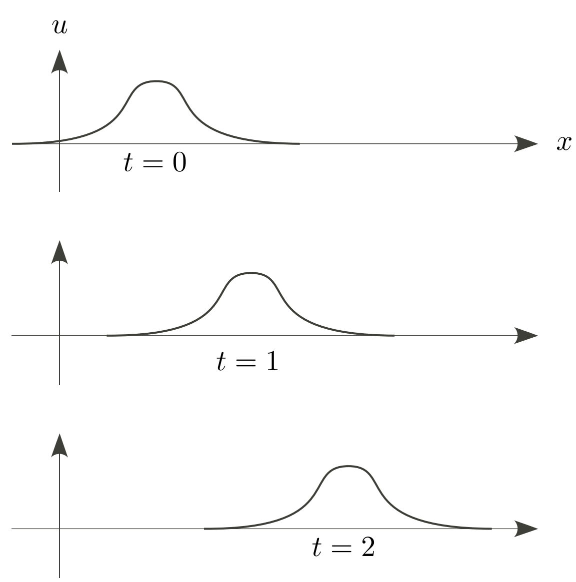
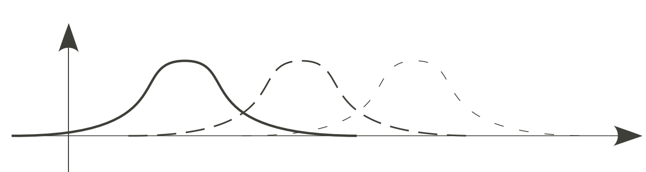



Figure 3: Transport snapshots

or all in one figure:



## 2 The Wave Equation

The one-dimensional wave equation

$$\boxed{u_{tt} = c^2 u_{xx}}$$

models the motion of vibrating strings, sound waves in a pipe, water waves in a canal, etc.

Why? Here is a rough argument for the case of a string. Assume the string is flexible, homogeneous (of constant density $\rho$), and stretched to a constant tension[†] $T$. Now set it in motion, e.g. by plucking it or hitting it with a hammer. Let $u(x,t)$ be its displacement from equilibrium at position $x$ and time $t$, so for fixed $t$, the graph in the $xu$-plane of $u = u(x,t)$ gives the shape of the string at time $t$ (Figure 4).

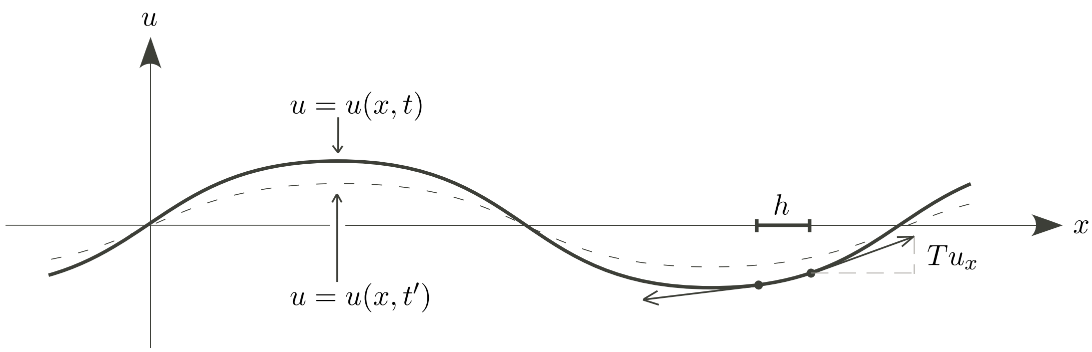



Figure 4: Vibrating string

Assuming the vibrations are transverse (vertical motion only) the force on a small segment $[x, x+h]$ of the string at time $t$ is approximately $Tu_x(x+h,t) - Tu_x(x,t)$ (the difference of the vertical components of the tensions at the ends; the horizontal components are $T$). This segment has mass $\rho h$ and acceleration $u_{tt}$, and so Newton's law $F = ma$ becomes

$$T(u_x(x+h,t) - u_x(x,t)) \approx \rho h u_{tt}.$$

Dividing by $h$ and taking the limit as $h \to 0$ gives $Tu_{xx} = \rho u_{tt}$, i.e.

$$u_{tt} = c^2 u_{xx}$$

where $c = \sqrt{T/\rho}$.

Remark This rough model ignores some forces that may affect the motion, such as

- external forces (e.g. gravity)
- friction (e.g. air resistance) – proportional to velocity
- restoring forces – proportional to displacement

which lead to a PDE

$$u_{tt} = c^2 u_{xx} - r u_t - k u + F(x,t)$$

for suitable positive constants $r$ and $k$, that is much harder to solve.

[†]The tension in the string at a point is the force exerted by the part of the string to one side of the point on the part to the other side, directed tangentially along the string because it is flexible.

**The Infinite String** (wave equation on the line $\mathbb{R}$)

It is physically evident that the motion of a string with a given initial position also depends on its initial velocity. This leads to the following initial value problem for the displacement $u(x,t)$ of the string:

$$\begin{cases} u_{tt} = c^2 u_{xx} & \text{for } -\infty < x < \infty \text{ and } t \geq 0 \\ u(x,0) = \phi(x),\ u_t(x,0) = \psi(x) & \text{for all } x \end{cases}$$

The functions $\phi$ and $\psi$ are called the initial position and initial velocity of the problem.

**d'Alembert's Formula** † (1746)

Without the initial conditions, the wave equation is easily solved by changing variables since the operator $\partial_{tt} - c^2\partial_{xx}$ factors as $(\partial_t - c\partial_x)(\partial_t + c\partial_x)$. In particular it reduces to $u_{rs} = 0$ under the oioi-substitution $r = x + ct$, $s = x - ct$ and so the general solution is

$$\boxed{u(x,t) = f(x+ct) + g(x-ct)}$$

where $f$ and $g$ are arbitrary $C^2$ (twice continuously differentiable) functions.

Geometric Interpretation The first term $f(x+ct)$ describes a wave of shape $u = f(x)$ traveling to the left at speed $c$, while $g(x-ct)$ is the wave $u = g(x)$ moving to the right at speed $c$. So the general solution is the superposition of two arbitrary waves moving in opposite directions, both at speed $c$ (the speed of propagation).

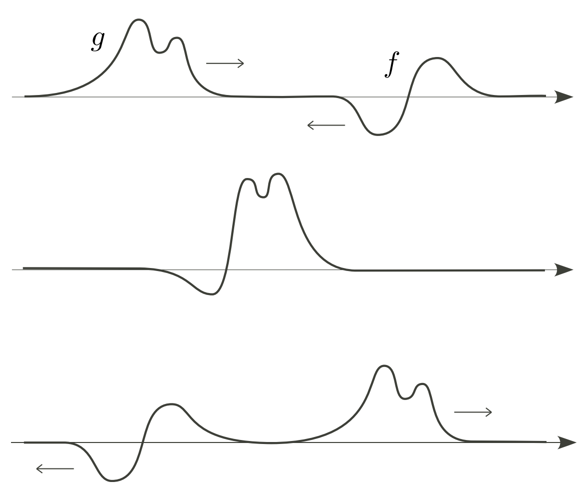



Figure 5: Passing waves

The lines $x \pm ct =$ constant (of slopes $\mp 1/c$) in the $xt$-plane are called the characteristics of the wave equation: $f(x+ct)$ is constant along those of negative slope while $g(x-ct)$ is constant along those of positive slope. These lines play an important role in the qualitative analysis of the wave equation (see the remarks on "causality" below).

† proved independently by Euler (1748)

Now we impose the initial conditions $u(x,0) = \phi(x)$ and $u_t(x,0) = \psi(x)$ on the general solution $u(x,t) = f(x+ct) + g(x-ct)$. They imply $u(x,0) = f(x) + g(x) = \phi(x)$ and $u_t(x,0) = cf'(x) - cg'(x) = \psi(x)$. Integrating the second equation gives $cf - cg = \Psi$, where $\Psi$ is an antiderivative of $\psi$. This gives two equations relating the unknowns $f$ and $g$:

$$f + g = \phi \qquad \text{and} \qquad f - g = \Psi/c.$$

Adding these equations gives $2f = \phi + \Psi/c$, and subtracting them gives $2g = \phi - \Psi/2$. Thus $f = \frac{1}{2}\phi + \frac{1}{2c}\Psi$ and $g = \frac{1}{2}\phi - \frac{1}{2c}\Psi$, and so

$$\boxed{u(x,t) = \frac{1}{2}\left(\phi(x+ct) + \phi(x-ct)\right) + \frac{1}{2c}\left(\Psi(x+ct) - \Psi(x-ct)\right)}$$

Here $\Psi$ can be <u>any</u> antiderivative of $\psi$ (they all differ by constants which cancel). This is called <u>d'Alembert's</u> <u>formula</u>. It can also be written in integral form

$$u(x,t) = \boxed{\frac{\phi(x+ct) + \phi(x-ct)}{2} + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(s)ds}$$

using the fundamental theorem of calculus. This is the version given in Strauss.

**<u>Example</u>** The initial value problem $u_{tt} = 4u_{xx}$, $u(x,0) = 2\sin 3x$, $u_t(x,0) = 6x^2$ has solution $u(x,t) = \sin 3(x+2t) + \sin 3(x-2t) + 6x^2t + 8t^3$.

## Consequences of d'Alembert's Formula

### Existence and Uniqueness of Solutions

**<u>Theorem</u>** *If $\phi$ is $C^2$ and $\psi$ is $C^1$, then the initial value problem for the wave equation has a unique $C^2$ solution given by d'Alembert's formula.*

<u>Proof</u> $\psi$ is $C^1 \Longrightarrow \Psi$ is $C^2 \Longrightarrow u$ is $C^2$. Thus $u$ is a bona fide solution to the IVP. $\square$

The uniqueness of solutions has a particularly important consequence for the analysis of waves with boundary conditions (below):

**<u>Corollary</u>** (Parity preservation) *If $\phi$ and $\psi$ are both even or both odd,*[†] *then the solution $u(x,t)$ is correspondingly even or odd as a function of $x$ (for any fixed $t$).*

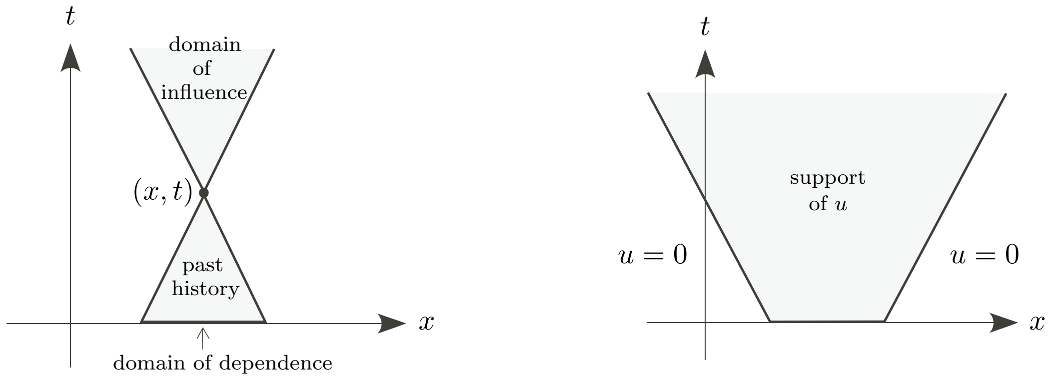
<u>Proof</u> If $\phi$ and $\psi$ are even, set $v(x,t) = u(-x,t)$. An exercise in the chain rule shows that $v$ satisfies the same IVP: $v_{tt} = u_{tt}(-x,t) = c^2v_{xx}$, $v(x,0) = u(-x,0) = \phi(-x) = \phi(x)$ and $v_t(x,0) = u_t(-x,0) = \psi(-x) = \psi(x)$. By uniqueness $v(x,t) = u(x,t)$, which says that $u(x,t)$ is even in $x$. The odd case is similar, starting with $v(x,t) = -u(-x,t)$. $\square$

<u>Remark</u> This can also be proved directly from d'Alembert's formula, noting that any function that is odd or even has an antiderivative of the opposite parity.

[†]Recall that a function $f : \mathbb{R} \to \mathbb{R}$ is <u>even</u> if $f(-x) = f(x)$ for all $x$, and is <u>odd</u> if $f(-x) = -f(x)$ for all $x$.

Causality

By d'Alembert's formula, the value $u(x,t)$ at any point $(x,t)$ depends only on the values of $\phi$ at the two points $x \pm ct$ and the values of $\psi$ within the interval $[x-ct, x+ct]$ (or equivalently of $\Psi$ at $x \pm ct$). This interval on the $x$-axis is called the domain of dependence of $(x,t)$, and is the base of a triangle, with top vertex $(x,t)$, called the past history of $(x,t)$ (the other two sides of this triangle are characteristics). See Figure 11(a).




(a) past and future (b) compact support

Figure 6: Causality

Similarly the domain of influence of $(x,t)$ is the region lying above the two characteristics through $(x,t)$. It consists of all points in the plane at which the value of $u$ is affected in any way by $u(x,t)$ (Figure 11(a) again).

An important consequence of causality for waves is that

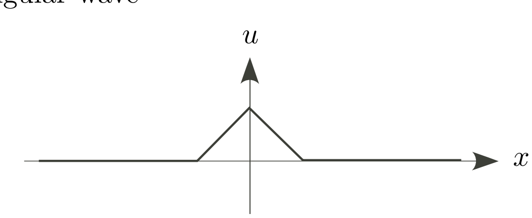
*local disturbances propagate with finite speed*

namely $c$. As an illustration, suppose that $\phi$ and $\psi$ vanish outside a finite interval $[a,b]$. Then $u$ vanishes outside the union of the domains of influence of all the points in $[a,b]$ (whose level $t$ crossection is the interval $[x-ct, x+ct]$; see Figure 11(b)). (For homework you are asked what more can be said if $\psi$ is zero everywhere. Hint: the domain of influence of a point on the $x$-axis is then just the two upward characteristic rays emanating from the point.)

Singularities

If $\phi$ and $\Psi$ are only piecewise $C^2$ (meaning $C^2$ except at a discrete set of "singular" points where they need not even be continuous) then the function $u$ given by d'Alembert's formula will satisfy the IVP except possibly along the characteristics through the singular points (on the $x$-axis). For example consider the "triangular wave"

$$\wedge(x) := \begin{cases} 1-|x| & |x|<1 \\ 0 & |x| \geq 1 \end{cases}$$



which is not differentiable at $\pm 1$ and 0. Then if $c = \frac{1}{4}$, $\phi = \wedge$ (as in a "three finger pluck") and $\psi = 0$ (so we can take $\Psi = 0$ as well) then the snapshots of the wave (in the $xu$-plane ) at one second intervals look like:

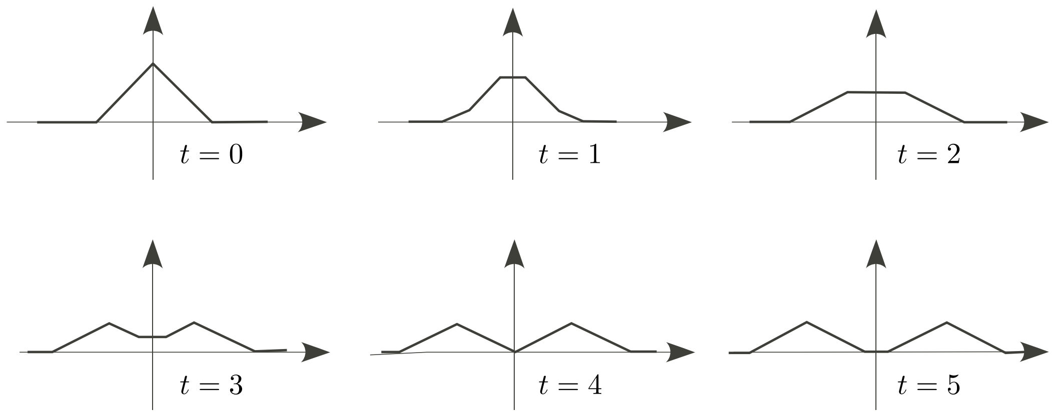



Figure 7: $\wedge$ snapshots

Note that the snapshot at each time $t$ is differentiable except at isolated points, namely the points at level $t$ in the $xt$-plane on the characteristics through the singularities in $\wedge$ as shown in Figure 8.

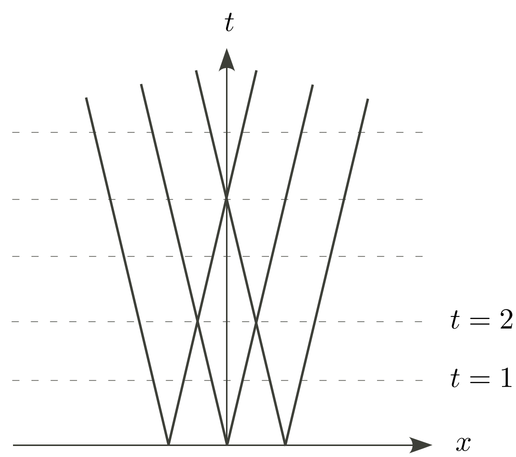



Figure 8: Propagating singularities

This illustrates the principle that

*singularities in the initial data are propagated along characteristics.*

This can be explained in general using causality. In the homework, you are asked to make a similar analysis using a "square wave" as the initial position or initial velocity (the latter models a hammer striking the string, as in a piano).

**The semi-infinite string** (wave equation on the half line $\mathbb{R}_+$)

Suppose that one end of the string is fixed. This leads to the following "Initial Boundary Value Problem" (IBVP) for the displacement $u(x,t)$:

$$\begin{cases} u_{tt} = c^2 u_{xx} & \text{for } x \geq 0 \text{ and } t \geq 0 \\ u(x,0) = \phi(x),\ u_t(x,0) = \psi(x) & \text{for all } x \\ u(0,t) = 0 & \text{for all } t \end{cases}$$

The restrictions in the second line are called the initial conditions (we assume that $\phi$ and $\psi$ are defined on $\mathbb{R}_+$ with $\phi(0) = \psi(0) = 0$) and the last restriction is the boundary condition.

To solve this, let $\phi_{\text{odd}}$ and $\psi_{\text{odd}}$ be the odd extensions of $\phi$ and $\psi$ to $\mathbb{R}$, i.e.

$$\phi_{\text{odd}}(x) = \begin{cases} \phi(x) & \text{for } x \geq 0 \\ -\psi(-x) & \text{for } x < 0 \end{cases}$$

and similarly for $\psi_{\text{odd}}$. Let $u(x,t)$ be the solution of the IVP on $\mathbb{R}$ with initial data $\phi_{\text{odd}}$ and $\psi_{\text{odd}}$. Then by parity preservation (proved above) $u(x,t)$ is odd in $x$, so $u(0,t) = 0$ for all $t$. Therefore the boundary condition is automatically satisfied, and so the restriction of $u(x,t)$ to $x \geq 0$ is the solution to the IBVP. By d'Alembert's formula

$$u(x,t) = f(x+ct) + g(x-ct)$$

where $f = \frac{1}{2}\phi_{\text{odd}} + \frac{1}{2c}\Psi_{\text{even}}$ and $g = \frac{1}{2}\phi_{\text{odd}} - \frac{1}{2c}\Psi_{\text{even}}$. Here $\Psi_{\text{even}}$ is any antiderivative of $\psi_{\text{odd}}$ on $\mathbb{R}$, or equivalently the even extension of any antiderivative $\Psi$ of $\psi$ on $\mathbb{R}_+$.

The solution can be expressed directly in terms of $\phi$ and $\psi$ as follows. First note that the parity subscripts in $f$ can be dropped since $x + ct \geq 0$. They can also be dropped for $g$ when $x - ct \geq 0$, but if $x - ct \leq 0$ then

$$g(x-ct) = -\tfrac{1}{2}\phi(ct-x) - \tfrac{1}{2c}\Psi(ct-x) = -f(ct-x).$$

Therefore the solution to the IBVP is

$$\boxed{u(x,t) = f(x+ct) + \begin{cases} g(x-ct) & \text{for } x \geq ct \\ -f(ct-x) & \text{for } x \leq ct \end{cases}}$$

where $f = \frac{1}{2}\phi + \frac{1}{2c}\Psi$ and $g = \frac{1}{2}\phi - \frac{1}{2c}\Psi$ (which are just the restrictions to $\mathbb{R}_+$ of the $f$ and $g$ above) or in integral form

$$\boxed{u(x,t) = \begin{cases} \dfrac{\phi(x+ct) + \phi(x-ct)}{2} + \dfrac{1}{2c}\displaystyle\int_{x-ct}^{x+ct} \psi(s)ds & \text{for } x \geq ct \\[2ex] \dfrac{\phi(ct+x) - \phi(ct-x)}{2} + \dfrac{1}{2c}\displaystyle\int_{ct-x}^{ct+x} \psi(s)ds & \text{for } x \leq ct \end{cases}}$$

Note that these formulas agree when $x = ct$ but that $u$ might be singular (i.e. not differentiable, or even not continuous) along the characteristic $x = ct$. We will also call this d'Alembert's Formula for the half line.

**Example** The wave equation $u_{tt} = 4u_{xx}$ on the half line with initial conditions $u(x, 0) = 2 \sin 3x$, $u_t(x, 0) = 6x^2$ and boundary condition $u(0, t) = 0$ has solution

$$u(x,t) = \sin 3(x+2t) + \sin 3(x-2t) + \begin{cases} 6x^2 t + 8t^3 & \text{for } x \geq 2t \\ x^3 + 12xt^2 & \text{for } x \leq 2t. \end{cases}$$

## Physical Interpretation: Waves reflect off the boundary

For simplicity, assume that $\phi$ and $\psi$ vanish outside $[a, b]$, and further that $\int_a^b \phi(s)ds = 0$, so $\psi$ has an antiderivative $\Psi$ that also vanishes outside $[a, b]$. Then d'Alembert's formula shows that for $t < a/c$, the wave is the superposition of two profiles traveling in opposite directions at speed $c$.

What happens at $t = a/c$ when the wave moving to the left reaches the boundary? The answer is that it temporarily changes shape (during the time interval $[a/c, b/c]$) and then returns to its original shape flipped over (i.e. rotated a half turn) and traveling back in the opposite direction.

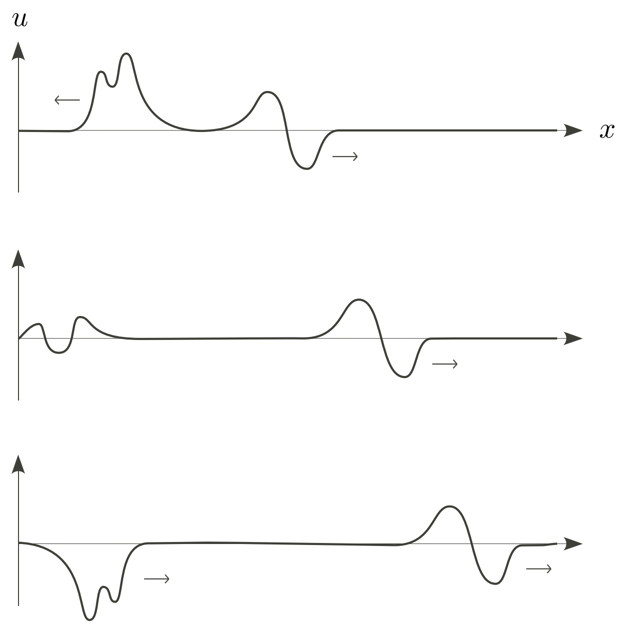



Figure 9: Wave reflection

This can be proved using d'Alembert's Formula above or by staring at the following figure showing the evolution of $\phi_{\text{odd}}$:

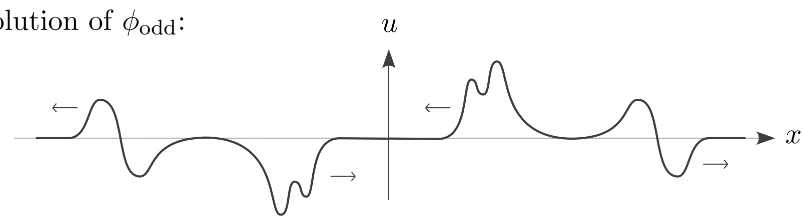
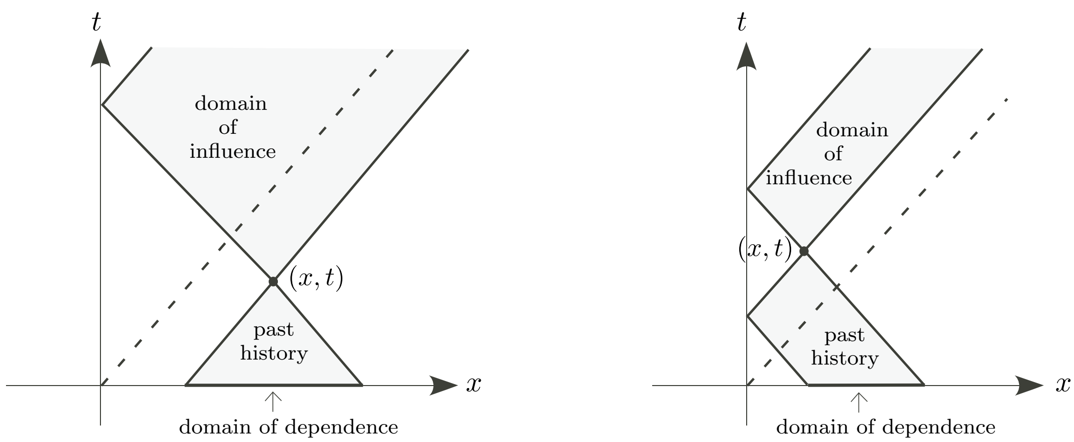



Figure 10: Why waves reflect

**Causality** (for the semi-infinite string)

D'Alembert's integral formula for the half-line leads to the following modified pictures for the domains of dependence and influence of a point $(x, t)$, depending upon whether $x - ct$ is positive or negative. The dotted line $x = ct$ separates the regions in which the two cases of the formula apply. In particular, the domain of dependence is $[x - ct, x + ct]$ in the first case (as with the infinite string), and $[ct - x, ct + x]$ in the second (due to the reflection).




(a) past and future (b) compact support

Figure 11: Causality on the half-line

In anticipation of the study of finite strings (where $x$ is restricted to lie in an interval $[0, \ell]$ with boundary conditions $u(0, t) = u(\ell, t) = 0$) you are encouraged to explore the corresponding picture for causality, as suggested by the following figure.

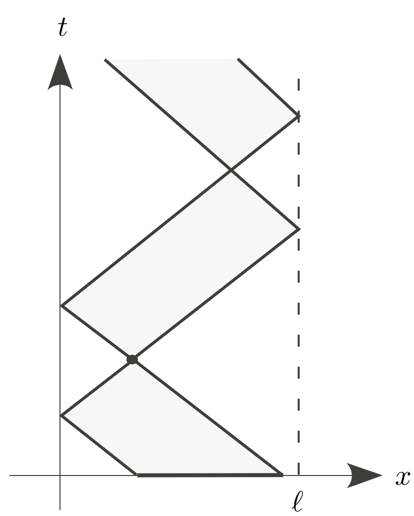



Figure 12: Causality in a finite interval

We will treat this later from a different point of view, using Fourier series, but it is instructive and of historical interest to reconcile the two perspectives (cf. Strauss pp. 61–64).

# 3 The Heat Equation

The one-dimensional heat equation (a.k.a. the diffusion equation)

$$\boxed{u_t = ku_{xx}}$$

models the flow of heat in a rod, the diffusion of a chemical in a tube of liquid, etc.

Why? The easiest situation to understand is a chemical, such as a dye, diffusing in a tube of motionless liquid. Let $u(x,t)$ be the concentration (mass / unit length) of dye at position $x$ and time $t$. Then the mass of dye in any given interval $[a,b]$ at a fixed time $t$ is

$$m = \int_a^b u(x,t)dx$$

Of course $m$ changes with time at a rate equal to net rate of flow of dye into the interval. By experimental observation, the dye flows from regions of higher to lower concentration at a rate proportional to its concentration gradient (Fick's law of diffusion) and so the net rate of flow is $dm/dt = ku_x(b,t) - ku_x(a,t)$ for some positive constant $k$.[†] But $dm/dt$ can also be computed by differentiating the integral expression for $m$ above (using Theorem 1 in the Appendix of Strauss to interchange the order of differentiation and integration) giving

$$\int_a^b u_t(x,t)dx = k(u_x(b,t) - u_x(a,t)) = \int_a^b ku_{xx}(x,t)dx$$

where the last equality follows from the fundamental theorem of calculus. This is true for all intervals $[a,b]$, so the integrands must be equal, which is the diffusion equation.

Similarly heat flows from hot to cold at a speed proportional to the temperature gradient (Fourier's law) so if $u(x,t)$ denotes the temperature of a rod at position $x$ and time $t$, the same argument (with heat content replacing mass) leads once again to the heat equation.

## Solving the unbounded heat equation

We first solve the heat equation on $\mathbb{R}$ (an infinite rod). Physical intuition tells us that there should be a unique solution once we specify an initial temperature $\phi(x)$ for all $x$. Thus the temperature function $u(x,t)$ is the solution to the initial value problem:

$$\begin{cases} u_t = ku_{xx} & \text{for } -\infty < x < \infty \text{ and } t > 0 \\ u(x,0) = \phi(x) & \text{for all } x \end{cases}$$

For theoretical reasons, we assume $\phi$ is absolutely integrable (meaning that $\int_{-\infty}^{\infty} |\phi|$ is finite) although the formulas we derive hold more generally (e.g. for constant $\phi$).

[†] To understand the signs, consider the possibilities. If $u_x(b,t) > 0$ then there is more dye to the right of $b$ than to the left, so the dye flows into the interval. If $u_x(a,t) > 0$ then the dye flows out of the interval.

This turns out to be much harder to solve than the wave equation. For fun we write down the solution right away, and then (after a remark) explain how it is derived. Here it is:

$$\boxed{u(x,t) = \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} e^{-(x-y)^2/4kt} \phi(y)\, dy}$$

We shall call this Fourier's Formula.

**Remark** The integral in Fourier's formula cannot generally be evaluated in closed form, but if $\phi$ is piecewise constant then it can be expressed in terms of the error function

$$\operatorname{Erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-p^2}\, dp$$

from statistics.[†] To see this, note that the substitution $p = (y-x)/\sqrt{4kt}$ leads to an alternative form of Fourier's formula:

$$\boxed{u(x,t) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-p^2} \phi(x + p\sqrt{4kt})\, dp}$$

Now consider the characteristic function $\chi_{(a,b)}(x)$ of the interval $(a,b)$, which by definition has value 1 for $x \in (a,b)$ and zero elsewhere (any piecewise constant function is a linear combination of such functions). If $\phi = \chi_{(a,b)}$, then the integral above reduces to

$$\frac{1}{\sqrt{\pi}} \int_{(a-x)/\sqrt{4kt}}^{(b-x)/\sqrt{4kt}} e^{-p^2}\, dp$$

which yields the solution

$$u(x,t) = \frac{1}{2}\left(\operatorname{Erf}(\frac{x-a}{\sqrt{4kt}}) - \operatorname{Erf}(\frac{x-b}{\sqrt{4kt}})\right) \qquad \text{(since Erf is odd)}$$

This formula can also be applied when $a$ or $b$ is infinite, since $\operatorname{Erf}(\pm\infty) = \pm 1$.[†]

Our derivation of Fourier's formula will use the notion of the Fourier transform, which arises in connection with Fourier integrals. Fourier integrals play the same role for absolutely integrable functions that Fourier series play for periodic functions. Although we will study Fourier series in some detail later, we describe the basic set up here:

---

[†]The integrand $e^{-p^2}$ in Erf is called a Gaussian. Gaussians are exponentials of quadratic functions. They are ubiquitous in the theory of probability and statistics (normal distributions), physics, etc. A calculus exercise shows that

$$\int_{-\infty}^{\infty} e^{-p^2}\, dp = \sqrt{\pi}$$

It follows that $\operatorname{Erf}(\pm\infty) = \pm 1$, where by definition $\operatorname{Erf}(\pm\infty)$ means $\lim_{x\to\pm\infty} \operatorname{Erf}(x)$.

## Fourier Series

Any continuous, piecewise smooth function $f : \mathbb{R} \to \mathbb{C}$ of period $2\pi$ can be represented as a Fourier series

$$\boxed{f(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty}(a_k \cos kx + b_k \sin kx)}$$

where the coefficients are computed using Euler's formulas:

$$a_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos kx\,dx \qquad \text{and} \qquad b_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin kx\,dx$$

It is convenient to write this series in complex form. Using Euler's identity

$$e^{i\theta} = \cos\theta + i\sin\theta$$

the $k$th term (a.k.a. the $k$th-harmonic of $f$) $a_k \cos kx + b_k \sin kx$ can be rewritten as

$$c_k e^{ikx} + c_{-k}e^{-ikx}$$

where $c_k = (a_k - ib_k)/2$ and $c_{-k} = (a_k + ib_k)/2$. Using Euler's formulas for $a_k$, $b_k$ this gives

$$\boxed{f(x) = \sum_{k=-\infty}^{\infty} c_k e^{ikx} \qquad \text{where} \qquad c_k = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-ikx}\,dx}$$

Convergence questions will be discussed later.†

**Remark** If the continuity assumption on $f$ is relaxed to piecewise continuity, and if $f$ has a jump discontinuity at $x_0$, then the series converges to the average of the left and right hand limits of $f(x)$ as $x$ approaches $x_0$.

Now suppose that $f$ has period $2\ell$ instead of $2\pi$. Then the function $g(x) = f(x\ell/\pi)$ has period $2\pi$, and its Fourier expansion $g(x) = \sum c_k e^{ikx}$ translates (substituting $x\pi/\ell$ for $x$ in the integral for $c_k$) into the following Fourier expansion for $f(x) = g(x\pi/\ell)$

$$\boxed{f(x) = \sum_{k=-\infty}^{\infty} c_k e^{ikx\pi/\ell} \qquad \text{where} \qquad c_k = \frac{1}{2\ell}\int_{-\ell}^{\ell} f(x)e^{-ikx\pi/\ell}\,dx}$$

Note that if $f$ is only defined on the interval $[-\ell, \ell]$, then this expansion is still valid for all $x$ in that interval (seen by considering the periodic extension of $f$ to the whole line).

## Fourier Integrals and the Fourier Transform

Any continuous, piecewise smooth function $f : \mathbb{R} \to \mathbb{C}$ which is absolutely integrable (and thus nonperiodic; we shall call such a function nice) has a Fourier integral representation

$$\boxed{f(x) = \int_{-\infty}^{\infty} c_\omega e^{i\omega x}\,d\omega \qquad \text{where} \qquad c_\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x)e^{-i\omega x}\,dx}$$

† It will be shown that the Fourier series of $f$ converges pointwise to $f(x)$ (i.e. substituting any number $x_0$ for $x$ yields a numerical series converging to $f(x_0)$) and that it is the only trigonometric series with that property.

This is seen by taking the limit as $\ell \to \infty$ of the Fourier series for $f$ on $[-\ell, \ell]$, described above. Indeed, setting $\omega_k = k\pi/\ell$ and $\Delta\omega = \pi/\ell$ (the difference between successive $\omega_k$'s) and then substituting the formulas for the $c_k$'s into the series gives

$$f(x) = \sum_{k=-\infty}^{\infty} \left( \frac{1}{2\pi} \int_{-\ell}^{\ell} f(y) e^{-i\omega_k y}\, dy \right) e^{i\omega_k x} \Delta\omega$$

for any $x \in [-\ell, \ell]$. The right hand side looks like a Riemann sum for the Fourier integral above, noting that $\Delta\omega \to 0$ as $\ell \to \infty$. We omit the estimates required to rigorously prove that it in fact converges to that integral.

There is a beautiful symmetry inherent in the Fourier integral representation of a nice function, which we formulate in terms of the Fourier transform and its inverse. (In general, a transform is a linear operator that maps functions to functions.)

**Definition** Let $f : \mathbb{R} \to \mathbb{C}$ be a nice (i.e. continuous, absolutely integrable, piecewise smooth) function. Then the functions

$$\hat{f}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-i\omega x}\, dx \qquad \text{and} \qquad \check{f}(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(\omega) e^{i\omega x}\, d\omega$$

are called the Fourier Transform and Inverse Fourier Transform of $f$, respectively. Note that $\hat{f}$ need not be real valued, even if $f$ is.

We also sometimes write $\mathcal{F}(f)$ for $\hat{f}$, and $\mathcal{F}^{-1}(f)$ for $\check{f}$. Fourier's integral representation of $f$ is simply the statement that $f = \mathcal{F}^{-1}\mathcal{F}(f)$, i.e. $\mathcal{F}$ and $\mathcal{F}^{-1}$ are inverse transforms.

**Remark** When the variable of $f$ is spacial or temporal, written $x$ or $t$, then the variable of its transform $\hat{f}$ can be interpreted as a measure of frequency, and so is written $\omega$ following common physical convention.

**Examples** ① The Fourier transform of $f(x) = e^{-|x|}$ is

$$\begin{aligned}\hat{f}(\omega) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-|x|} e^{-i\omega x}\, dx = \frac{1}{\sqrt{2\pi}} \left( \int_{-\infty}^{0} e^{x(1-i\omega)}\, dx + \int_{0}^{\infty} e^{-x(1+i\omega)}\, dx \right) \\ &= \frac{1}{\sqrt{2\pi}} \left( \left. \frac{e^{x(1-i\omega)}}{1-i\omega} \right|_{-\infty}^{0} - \left. \frac{e^{-x(1+i\omega)}}{1+i\omega} \right|_{0}^{\infty} \right) = \frac{1}{\sqrt{2\pi}} \left( \frac{1}{1-i\omega} - 0 + 0 + \frac{1}{1+i\omega} \right) = \frac{\sqrt{2/\pi}}{1+\omega^2}\end{aligned}$$

In this case both $f$ and $\hat{f}$ are real valued.

② As will be seen below, the Fourier transform of any Gaussian is another Gaussian. As a warm up we show (by completing the square) that $e^{-x^2/2}$ is its own transform:

$$\widehat{e^{-x^2/2}} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-x^2/2} e^{-i\omega x}\, dx = \frac{1}{\sqrt{2\pi}} e^{-\omega^2/2} \int_{-\infty}^{\infty} e^{-(x+i\omega)^2/2}\, dx$$

Substituting $p = (x + i\omega)/\sqrt{2}$ in the last integral gives $\sqrt{2}\int_{-\infty}^{\infty} e^{-p^2}\, dp = \sqrt{2\pi}$. Thus

$$\boxed{\widehat{e^{-x^2/2}} = e^{-\omega^2/2}}$$

**Properties of the Fourier Transform**

1. (Linearity) $\mathcal{F}(cf+g) = c\,\mathcal{F}(f)+\mathcal{F}(g)$ for any nice functions $f, g$ and any constant $c$. This is immediate from the linearity of the integral.

2. (Shifting) Let $f_c(x) = f(x-c)$.† Then

$$\boxed{\widehat{f_c}(\omega)} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x-c)e^{-i\omega x}\,dx = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(u)e^{-i\omega(u+c)}\,du = \boxed{e^{-i\omega c}\hat{f}(\omega)}.$$

3. (Scaling) Let $f^c(x) = f(cx)$.† Then

$$\boxed{\widehat{f^c}(\omega)} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(cx)e^{-i\omega x}\,dx = \frac{1}{|c|}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(u)e^{-i\omega u/c}\,du = \boxed{\frac{1}{|c|}\hat{f}(\frac{\omega}{c})}$$

4. (Gaussians) As an application of scaling (with $f(x) = e^{-x^2/2}$ as in example 2 above, and $c = \sqrt{2a}$) we have

$$\boxed{\mathcal{F}(e^{-ax^2}) = \frac{1}{\sqrt{2a}}e^{-\omega^2/4a}} \qquad \text{and} \qquad \boxed{\mathcal{F}^{-1}(e^{-a\omega^2}) = \frac{1}{\sqrt{2a}}e^{-x^2/4a}}$$

for any constant $a$, where the second formula follows from the first by replacing $a$ by $1/4a$ and applying the inverse transform. Similar formulas hold for any Gaussian.

5. (Differentiation) Suppose that $f$ is differentiable, and that both $f$ and $f'$ are nice (in particular absolutely integrable). Then

$$\boxed{\widehat{f'}(\omega) = i\omega\hat{f}(\omega)}$$

To see this, integrate by parts:

$$\widehat{f'}(\omega) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-i\omega x}f'(x)\,dx = \frac{1}{\sqrt{2\pi}}e^{-i\omega x}f(x)\Big|_{-\infty}^{\infty} + \frac{i\omega}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-i\omega x}f(x)\,dx.$$

The first term on the right is zero since $|e^{i\omega x}| = 1$ and $\lim_{x\to\pm\infty} f(x) = 0$ (because $f$ is absolutely integrable) and the last term is $i\omega\hat{f}(\omega)$. Thus the Fourier transform converts differential equations into algebraic equations!

6. (Convolution) The convolution of two nice functions $f$ and $g$ is the function defined by

$$f*g\,(x) := \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x-y)g(y)\,dy$$

†If $c>0$ then the graph of $f_c$ is obtained by shifting the graph of $f$ to the right by a distance $c$.

†Then the graph of $f^c$ is obtained by horizontally scaling the graph of $f$ by a factor of $c$, squeezing it in if $c>1$ and spreading it out if $c<1$ (and reflecting through the vertical coordinate axis if $c<0$).

It turns out that the Fourier transform of a product of two functions is the convolution of their product: $\widehat{fg} = \hat{f} * \hat{g}$ . Indeed

$$\begin{aligned}\widehat{fg}(\omega) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)g(x)e^{-i\omega x}\,dx = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x)\left(\int_{-\infty}^{\infty}\hat{g}(\sigma)e^{i\sigma x}\,d\sigma\right)e^{-i\omega x}dx \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} f(x)e^{-i(\omega-\sigma)x}\,dx\right)\hat{g}(\sigma)d\sigma \quad \text{(reversing the order of integration)} \\ &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\hat{f}(\omega-\sigma)\hat{g}(\sigma)d\sigma = \hat{f} * \hat{g}\,(\omega)\end{aligned}$$

Similarly $\widehat{f * g} = \hat{f}\hat{g}$ by essentially the same calculation. Thus $\mathcal{F}$ converts products into convolutions and convolutions into products, and the same is true of the $\mathcal{F}^{-1}$.

Application to filtering in image/audio processing: Multiplying the Fourier transform of the data function by an appropriate step function to cut off high frequencies corresponds to convolving the data function with the inverse transform of the step function.

## **Derivation of Fourier's Formula** (using the Fourier transform)

We seek a solution $u(x,t)$ to the heat equation $u_t = ku_{xx}$ with initial temperature $u(x,0) = \phi(x)$. Consider the function $\hat{u}$ of two variables given by

$$\hat{u}(\omega,t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} u(x,t)e^{-i\omega t}\,dx.$$

For any fixed $t$, this is just the Fourier transform of the function $u(x,t)$, viewed as a function of $x$. Observe that $\widehat{u_t} = \hat{u}_t$ (by differentiating under the integral sign) and $\widehat{u_{xx}} = -\omega^2\hat{u}$ (by the differentiation property 5 above). Thus $u_t = ku_{xx}$ becomes a family of ODEs

$$\hat{u}_t = -k\omega^2\hat{u}$$

one for each frequency $\omega$. Imposing the initial condition $u(x,0) = \phi(x)$, or equivalently $\hat{u}(\omega,0) = \hat{\phi}(\omega)$, these have a uniform solution

$$\boxed{\hat{u}(\omega,t) = \hat{\phi}(\omega)e^{-k\omega^2 t}}$$

This is the solution in the "frequency domain". To rewrite it in the "spatial domain" we take the inverse transform, which by Property 6 above is the convolution $\phi(x)$ with the inverse transform of $e^{-k\omega^2 t}$, denoted $S(x,t)$. Since $e^{-k\omega^2 t}$ is Gaussian in $\omega$, $S(x,t)$ is Gaussian in $x$:

$$S(x,t) = \mathcal{F}^{-1}\left(e^{-k\omega^2 t}\right) = \frac{1}{\sqrt{2kt}}e^{-x^2/4kt}$$

(computed for example using Property 4 above with $a = kt$) and so

$$u(x,t) = S(x,t) * \phi(x) = \frac{1}{\sqrt{4\pi kt}}\int_{-\infty}^{\infty} e^{-(x-y)^2/4kt}\phi(y)\,dy$$

This is Fourier's formula.

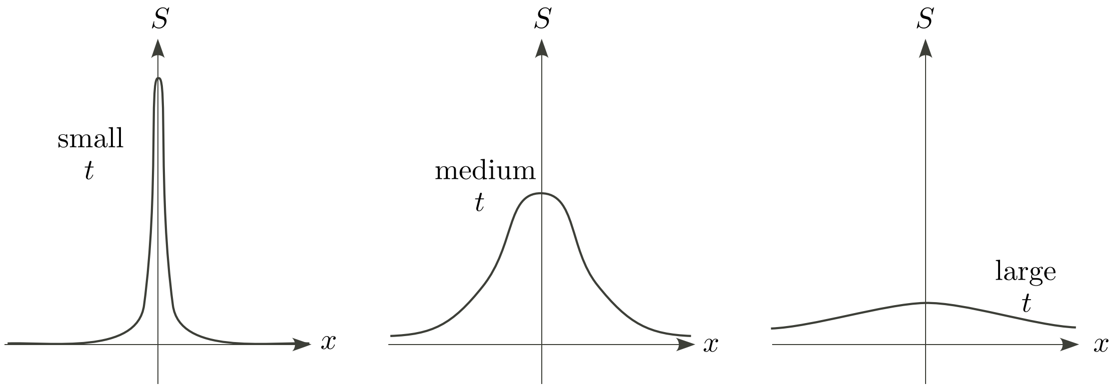



Figure 13: Heat Kernel

**Remark** The Gaussian $S(x,t)$ is called the heat kernel, a.k.a. the source or Green's function for the heat equation. Below are sketches of the graph of $S(x,t)$ as functions of $x$, for various values of $t$.

Note that $S(x,t) \to 0$ uniformly in $x$ as $t \to \infty$, and it follows that $u(x,t) \to 0$ as well. Convolving with $S(x,t)$ smoothes and spreads $\phi$ out more and more as time passes.

## Applications of Fourier's formula

### Existence and Uniqueness of Solutions

**Theorem** *If $\phi$ is continuous, piecewise smooth and absolutely integrable, then the heat equation $u_t = ku_{xx}$ with initial condition $u(x,0) = \phi(x)$ has a unique $C^\infty$ solution for $t > 0$, given by Fourier's formula $u(x,t) = S(x,t) * \phi(x)$ (where $S(x,t) = (1/\sqrt{2kt})\, e^{-x^2/4kt}$).*

Proof This follows from the fact that $S(x,t)$ is $C^\infty$ (in both variables).† □

As with the initial value problem for the wave equation, the uniqueness of solutions has the following useful consequence:

**Corollary** (Parity preservation) *If $\phi$ is odd or even, then the solution $u(x,t)$ is correspondingly odd or even as a function of $x$ (for any fixed $t$).*

Proof We treat odd case, leaving the even one to the reader. Set $v(x,t) = -u(-x,t)$. An exercise in the chain rule shows that $v$ satisfies the same IVP: $v_t = -u_t(-x,t) = kv_{xx}$ and $v(x,0) = -u(-x,0) = -\phi(-x) = \phi(x)$. By uniqueness $v(x,t) = u(x,t)$, which says that $u(x,t)$ is odd in $x$. □

### Propagation of local disturbances and singularities

There is no analogue for the heat equation of the wave causality principle. Consequently the principles of finite speed propagation of local disturbances and singularities for waves do not apply to heat flow. Indeed Fourier's formula predicts very different behavior, namely

*local disturbances propagate with infinite speed*

†Note that the convolution of any nice function with a $C^\infty$ function is $C^\infty$. For the theoretical details, see §3.5 in Strauss.

i.e. they are felt immediately at all distances. Perhaps this is physically unrealistic, but it is a mathematical consequence of our simple model. Furthermore

*singularities in the initial data immediately disappear*

since convolving with the $C^\infty$ heat kernel smoothes out the solution.

**The semi-infinite rod** (heat equation on the half line $\mathbb{R}_+$)

Suppose that one end of a heated rod is kept at a constant temperature. This leads to the following IBVP for the temperature $u(x,t)$:

$$\begin{cases} u_t = ku_{xx} & \text{for } x \geq 0 \text{ and } t > 0 \\ u(x,0) = \phi(x) & \text{for all } x \\ u(0,t) = 0 & \text{for all } t. \end{cases}$$

This is solved using the same trick as for the wave equation: Extend $\phi$ to an odd function $\phi_{\mathrm{odd}}$ on $\mathbb{R}$. Then the restriction to $x > 0$ of the solution to the heat equation on the whole line, with initial condition $u(x,0) = \phi_{\mathrm{odd}}(x)$, is the desired solution to the IBVP on the half line: it automatically satisfies the boundary condition by parity preservation. Thus by Fourier's formula the solution is

$$u(x,t) = \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} e^{-(x-y)^2/4kt} \phi_{\mathrm{odd}}(y)\, dy$$

To write this in terms of the original function $\phi$, split the integral into two: from $-\infty$ to 0 and from 0 to $\infty$. For the first we have

$$\int_{-\infty}^{0} e^{-(x-y)^2/4kt} \phi_{\mathrm{odd}}(y)\, dy = -\int_{0}^{\infty} e^{-(x+y)^2/4kt} \phi(y)\, dy$$

by substituting $-y$ for $y$ and using the oddness of $\phi_{\mathrm{odd}}$, and for the second we can simply drop the odd subscript on $\phi$. Thus the solution to the IBVP is

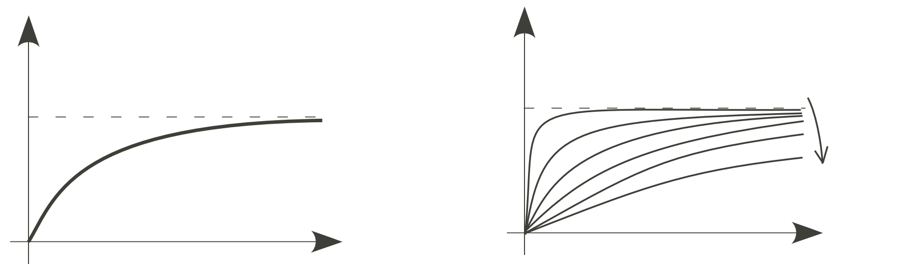
$$\boxed{u(x,t) = \frac{1}{\sqrt{4\pi kt}} \int_{0}^{\infty} \left( e^{-(x-y)^2/4kt} - e^{-(x+y)^2/4kt} \right) \phi(y)\, dy}$$

**Remark** The substitution $p = (y-x)/\sqrt{4kt}$ in the first exponential and $p = (y+x)/\sqrt{4kt}$ in the second leads to an alternative form of the formula

$$\boxed{u(x,t) = \frac{1}{\sqrt{\pi}} \left( \int_{-x/\sqrt{4kt}}^{\infty} e^{-p^2} \phi(p\sqrt{4kt} + x)\, dp - \int_{x/\sqrt{4kt}}^{\infty} e^{-p^2} \phi(p\sqrt{4kt} - x)\, dp \right)}$$

which is sometimes useful. For example, if $\phi$ is the characteristic function $\chi_{(a,b)}$ then this yields the solution

$$u(x,t) = \frac{1}{2} \left[ \left( \mathrm{Erf}(\frac{x-a}{\sqrt{4kt}}) - \mathrm{Erf}(\frac{x-b}{\sqrt{4kt}}) \right) + \left( \mathrm{Erf}(\frac{x+a}{\sqrt{4kt}}) - \mathrm{Erf}(\frac{x+b}{\sqrt{4kt}}) \right) \right].$$




(a) $\mathrm{Erf}(x)$ (b) $\mathrm{Erf}(x/\sqrt{4kt})$

Figure 14: Constant initial temperature

In particular, if $\phi = 1 = \chi_{(0,\infty)}$ then the solution is

$$u(x,t) = \mathrm{Erf}(\frac{x}{\sqrt{4kt}}).$$

Rescaling the graph of Erf in Figure 14(a)) gives the graphs of $u(x,t)$ for various values of $t$, as in Figure 14(b)).

# 4 Sources

In this short section, we derive solutions to the <u>inhomogeneous</u> wave and heat equations on the line.† These look like the corresponding homogeneous solutions (d'Alembert's and Fourier's formulas) but with one additional term accounting for the inhomogeneity.

*Wave equation with a source*

The IVP for the inhomogeneous wave equation on the line is

$$\begin{cases} u_{tt} = c^2 u_{xx} + f(x,t) & \text{for } -\infty < x < \infty,\ t \geq 0 \\ u(x,0) = \phi(x) \text{ and } u_t(x,0) = \psi(x) & \text{for all } x \end{cases} \tag{1}$$

where $f(x,t)$ should be viewed as an external force applied to the string, possibly varying with time. Here is the solution:

$$\boxed{u(x,t) = \frac{1}{2}(\phi(x+ct) + \phi(x-ct)) + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi + \frac{1}{2c}\iint_{\triangle} f}$$

where $\triangle$ is the "past history" triangle of $(x,t)$ in the $xt$-plane, with vertices $v_1 = (x-ct, 0)$, $v_2 = (x+ct, 0)$ and $v_3 = (x,t)$.

Why is this so? Among the many approaches (see §3.4 in Strauss) perhaps the easiest is simply to appeal to Green's Theorem, from Multivariable Calculus:

†For the case of the half line with various boundary conditions, see Strauss p. 76 and pp. 67–68.

**Green's Theorem** *If $P, Q : \mathbb{R}^2 \to \mathbb{R}$ are $C^1$ functions and $D$ is a region in $\mathbb{R}^2$ bounded by a piecewise smooth simple closed curve $\partial D$, then (viewing $\mathbb{R}^2$ as the $xt$-plane)*

$$\int_{\partial D} P\,dx + Q\,dt \;=\; \iint_D (Q_x - P_t)\,dx\,dt\;.$$

*Here $\partial D$ is by convention oriented positively (i.e. counterclockwise).*

To obtain the formula for the solution $u(x,t)$ to (3) we apply Green's Theorem with

$$P(x,t) = u_t(x,t) \qquad \text{and} \qquad Q(x,t) = c^2 u_x(x,t)$$

and $D = \triangle$. Note that $\partial\triangle$ consists of three oriented edges $C_1 = \overrightarrow{v_1v_2}$, $C_2 = \overrightarrow{v_2v_3}$ and $C_3 = \overrightarrow{v_3v_1}$. Also $f(x,t) = u_{tt} - c^2u_{xx} = P_t - Q_x$ and so

$$\iint_\triangle f \;=\; \iint_\triangle (P_t - Q_x)\,dx\,dt \;=\; -\int_{\partial\triangle} P\,dx + Q\,dt \;=\; -(I_1 + I_2 + I_3)$$

where

$$I_k = \int_{C_k} P\,dx + Q\,dt = \int_{C_k} u_t\,dx + c^2u_x\,dt.$$

Now along $C_1$ we have $dt = 0$ and $u_t = \phi$, and so $I_1 = \displaystyle\int_{x-ct}^{x+ct} \psi$.

Along $C_2$, which is part of a slope $-1/c$ characteristic line of the differential equation, $x+ct$ is constant and so $dx + cdt = 0 \Longrightarrow u_tdx + c^2u_xdt = -cu_tdt - cu_xdx = -cdu$. Therefore

$$I_2 = -c\int_{C_k} du = -c(u(v_3) - u(v_2)) = c(\phi(x+ct) - u(x,t))$$

by the fundamental theorem of calculus, and similarly $I_3 = c(\phi(x-ct) - u(x,t))$.

Putting these calculations together gives

$$\iint_\triangle f \;=\; -\int_{x-ct}^{x+ct} \psi \;-\; c(\phi(x+ct) + \phi(x-ct)) \;+\; 2cu(x,t)$$

which upon dividing by $2c$ and rearranging terms yields the desired formula.

**Example** The wave equation $u_{tt} = u_{xx} + xt$ with initial conditions $u(x,0) = 0 = u_t(x,0)$ has solution

$$u(x,t) \;=\; \frac{1}{2}\iint_\triangle ys\,dy\,ds \;=\; \frac{1}{2}\int_0^t \left(\int_{x-(t-s)}^{x+(t-s)} y\,dy\right) s\,ds$$

$$=\; \int_0^t x(t-s)s\,ds \;=\; xt^3/6.$$

*Heat equation with a source*

The IVP for the inhomogeneous heat equation on the line is

$$\begin{cases} u_t = ku_{xx} + f(x,t) & \text{for } -\infty < x < \infty,\ t > 0 \\ u(x,0) = \phi(x) & \text{for all } x \end{cases} \tag{2}$$

where $f(x,t)$ should be thought of as a heat source or sink. Here is the solution:

$$\boxed{\begin{aligned} u(x,t) &= \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} e^{-(x-y)^2/4kt} \phi(y)\, dy \\ &\quad + \int_0^t \frac{1}{\sqrt{4\pi k(t-s)}} \int_{-\infty}^{\infty} e^{-(x-y)^2/4k(t-s)} f(y,s)\, dy\, ds \end{aligned}}$$

or written in terms of convolutions (with respect to $x$)

$$u(x,t) = S(x,t) * \phi(x) + \int_0^t S(x,t-s) * f(x,s)\, ds$$

where $S(x,t) = (1/\sqrt{2kt})e^{-x^2/4kt}$ is the heat kernel. In physical terms this says that the effect of the source on the temperature $u(x,t)$ is the superposition of the impulses it provides at all the times $s < t$.

Now why is this the correct formula? First observe that the solution to (1) is clearly the sum of the solution $u(x,t) = S(x,t) * \phi(x)$ to the associated homogeneous problem

$$\begin{cases} u_t = ku_{xx} \\ u(x,0) = \phi(x) \end{cases}$$

and the solution $v(x,t)$ to the special case of (1) with $\phi(x) = 0$:

$$\begin{cases} v_t = kv_{xx} + f(x,t) \\ v(x,0) = 0 \end{cases} \tag{3}$$

Thus it remains to show that $v(x,t) = \int_0^t S(x,t-s) * f(x,s)\, ds$.

This is an instance of Duhamel's Principle, which asserts in general that the solution to an inhomogeneous PDE with homogeneous initial conditions is a superposition of solutions to the associated homogeneous PDE with various initial conditions specified by the inhomogeneous term in the original PDE. For the case at hand:

**Claim** *The solution $v(x,t)$ to (2) is a superposition of solutions $v^s(x,t) = S(x,t) * f(x,s)$ (the convolution is with respect to $x$) to the family of associated homogeneous problems*

$$\begin{cases} v^s_t = kv^s_{xx} \\ v^s(x,0) = f(x,s) \end{cases} \tag{$s$}$$

*for $s < t$. (Note that the superscript $s$ parametrizes the family; it is not an exponent.) In particular $v(x,t) = \int_0^t v^s(x,t-s)\, ds$.*

To establish the claim we verify that $v(x,t)$ satisfies (3). Taking the partial derivative with respect to $t$ is tricky, however, since $v(x,t)$ is defined by an integral in which $t$ appears both in the integration bound and in the integrand, so we appeal to the following general result (cf. Strauss' Theorem 3 in Appendix A.3).

**Theorem** *Consider the integral* $I(t) = \int_{a(t)}^{b(t)} g(s,t)\, ds$ *where* $a, b : \mathbb{R} \to \mathbb{R}$ *and* $g : \mathbb{R}^2 \to \mathbb{R}$ *are* $C^1$ *functions. Then*

$$I'(t) = \int_{a(t)}^{b(t)} g_t(s,t)\, ds + g(b(t),t)b'(t) - g(a(t),t)a'(t)$$

*where the* $g_t$ *denotes* $\partial g/\partial t$*, as usual.*

Proof Set $J(a,b,t) = \int_a^b g(s,t)\, ds$, where $a$, $b$ and $t$ are independent variables (so $I(t) = J(a(t), b(t), t)$). Then by the chain rule and the fundamental theorem of calculus, we compute $I'(t) = J_a a'(t) + J_b b'(t) + J_t = -g(a(t),t)a'(t) + f(b(t),t)b'(t) + \int_{a(t)}^{b(t)} g_t(s,t)\, ds$, as stated. □

Using this result we compute

$$\begin{aligned} v_t(x,t) &= \int_0^t v_t^s(x, t-s)\, ds + v^t(x, t-t) \\ &= \int_0^t k v_{xx}^s(x, t-s)\, ds + f(x,t) = k v_{xx}(x,t) + f(x,t) \end{aligned}$$

and clearly $v(x,0) = 0$. This completes the proof of the claim, and thus establishes the formula for the solution to (2).

## 5 Boundary Value Problems

Partial differential equations typically have many solutions. The specification of auxiliary conditions, however, often singles one out. Up until this point we have dealt primarily with initial conditions, although boundary conditions have arisen in our discussion of the wave and heat equations on the half line.

Boundary conditions have also arisen briefly in our discussion of the wave and heat equation on the half line, and also arise in the physically realistic situation of PDE's on a bounded domain. In particular, for functions $u(x,t)$ of two variables (space and time) on a half line $[a, \infty)$ or on a finite interval $[a, b]$, a boundary condition is a restriction on $u$ at the endpoints $a$ (or $a$ and $b$) for all times $t$. The three most important types are where

- $u$ is specified (Dirichlet boundary condition)
- $u_x$ is specified (Nemann boundary condition)
- some linear combination of $u$ and $u_x$ is specified (Robin boundary condition)

If the condition specifies the value to be zero, then it is said to be a homogeneous condition.

DID NOT TEX NOTES FOR THE REST OF THIS SECTION AND MOST OF THE NEXT ...

# 6 Fourier Series

Under construction, but here is a rough form of the last lecture in this section:

Let $\phi : \mathbb{R} \to \mathbb{R}$ be a $2\pi$-periodic function.

**Last time**

Proved that $\phi$ is $C^1 \Longrightarrow$ the Fourier series of $\phi$ converges pointwise to $\phi$, i.e. for fixed $x$,

$$\phi(x) = \frac{1}{2}A_0 + \sum_{n=1}^{\infty} A_n \cos nx + B_n \sin nx \qquad \text{(meaning RHS} \to \phi(x)) \qquad (*)$$

where (Euler's formulas)

$$A_n = \frac{1}{\pi}\int_{-\pi}^{\pi} \phi(x)\cos nx\, dx \qquad \text{and} \qquad B_n = \frac{1}{\pi}\int_{-\pi}^{\pi} \phi(x)\sin nx\, dx$$

**Today: Uniform convergence** Recall

1. The series $\sum_{n=1}^{\infty} f_n(x)$ converges uniformly to $f(x)$ means the sequence of partial sums $s_N(x) = \sum_{n=1}^{N} f_n(x)$ converges uniformly to $f(x)$, i.e. given $\varepsilon > 0$, $\exists M$ such that

   $$N > M \Longrightarrow |s_N(x) - f(x)| < \varepsilon$$

   for all $x$ (stronger than pointwise convergence, where $M$ in general depends on $x$).
2. The Weierstrass $M$-test: If $\exists$ constants $M_n$ such that $|f_n(x)| < M_n$ for all $x$ and $\sum M_n$ converges, then $\sum f_n(x)$ converges uniformly. (proof: use the Cauchy criterion)

**Theorem** *If $\phi$ is $C^2$, then the Fourier series of $\phi$ converges uniformly.*†

Proof Integrating Euler's formula for $A_n$ twice by parts gives

$$\begin{aligned} A_n &= \frac{1}{\pi}\left(\frac{1}{n}\phi(x)\sin(nx)\Big|_{-\pi}^{\pi} - \frac{1}{n}\int_{-\pi}^{\pi}\phi'(x)\sin(nx)\right) \\ &= -\frac{1}{n\pi}\left(\frac{1}{n}\phi'(x)\cos(nx)\Big|_{-\pi}^{\pi} + \frac{1}{n}\int_{-\pi}^{\pi}\phi''(x)\cos(nx)\right) \\ &= -\frac{1}{n^2\pi}\int_{-\pi}^{\pi}\phi''(x)\cos(nx) \end{aligned}$$

where the first term on the right hand side in the first and second lines vanish because of the $2\pi$-periodicity of $\phi$, $\phi'$, sin and cos. Since $\phi''$ and cos are bounded, we have $|A_n| \le A/n^2$ for some constant $A$, and similarly $|B_n| \le B/n^2$. Therefore

$$|A_n \cos nx + B_n \sin nx| \le \frac{A+B}{n^2}.$$

† to $\phi$, by the previous theorem

Since $\sum 1/n^2 < \infty$, the Weierstrass $M$-test shows that the convergence in $(*)$ is uniform. $\square$

**Theory application: Series solutions to PDEs**

Recall (without proof) the following basic result from real analysis:

**Lemma** *If the functions $f_n$ are differentiable on $[a,b]$, and series $\sum f_n$ and $\sum f_n'$ both converge uniformly (say to functions $f$ and $g$ respectively), then $(\sum f_n)' = \sum f_n'$ (meaning $f'(x) = g(x)$).*

Using this lemma, we can finally justify the series solutions previously given for the wave and heat equations on a finite interval. For example, for the heat equation on $[0,\pi]$ with Neumann boundary conditions, we have:

**Theorem** *Let $\phi$ be a continuous function on $[0,\pi]$ with Fourier cosine series coefficients*

$$A_n = \frac{2}{\pi}\int_0^\pi \phi(x)\cos nx\,dx.$$

*Then for each $t > 0$, the series*

$$u(x,t) = \tfrac{1}{2}A_0 + \textstyle\sum_{n=1}^\infty A_n e^{-n^2kt}\cos nx$$

*converges uniformly, and satisfies the heat equation $u_t = ku_{xx}$ with initial condition $\phi(x)$ (i.e. $u(x,t) \to \phi(x)$ as $t \to 0$) and Neumann boundary conditions $u_x(x,0) = u_x(x,\pi) = 0$. Furthermore $u(x,t) \to \frac{1}{2}A_0$ as $t \to \infty$.*

Proof First note that the $A_n$ are uniformly bounded (independent of $n$) since $\phi$ is bounded on $[0,\pi]$. Now set $f_n(x,t) = A_n e^{-n^2kt}\cos nx$, the $n$th term in the series $u(x,t)$, and temporarily fix a small $s > 0$.

We claim that $u(x,t) = \sum f_n$ converges uniformly in $x$ for any fixed $t$, and uniformly in $t > s$ for fixed $x$: Apply the Weierstrass $M$-test, noting that $\sum e^{-n^2ks} < \infty$ and

$$|f_n(x,t)| = |A_n e^{-n^2kt}\cos nx| \le Ae^{-n^2ks}$$

for some constant $A$. Similarly the series $\sum f_{n;t}$, $\sum f_{n;x}$ and $\sum f_{n;xx}$[†] converge uniformly, since $|nf_n| \le Bne^{-n^2ks}$ and $|n^2 f_n| \le Cn^2e^{-n^2ks}$, while $\sum ne^{-n^2ks}$ and $\sum n^2e^{-n^2ks}$ converge.

Thus, by the lemma we can differentiate term by term to obtain $u_t = u_{xx}$. The argument for the limiting behavior as $t \to 0$ and $t \to \infty$ is left to the reader. $\square$

Similarly for the wave equation with Dirichlet boundary conditions, can prove:

**Theorem** *Let $\phi$ be a $C^2$ function on $[0,\pi]$ with Fourier sine series coefficients*

$$B_n = \frac{2}{\pi}\int_0^\pi \phi(x)\sin nx\,dx.$$

[†]where $f_{n;t} = \partial f_n/\partial t$, etc.

*Then the wave equation $u_{tt} = c^2 u_{xx}$ with initial position $\phi$ and initial velocity zero, and boundary conditions $u(x,0) = 0 = u(\pi,0)$, has a unique solution given by the uniformly convegent series*

$$u(x,t) = \sum_{n=1}^{\infty} B_n \cos nct \sin nx = \frac{1}{2}(\phi(x+ct) + \phi(x-ct))$$

*where $\phi$ is the odd $2\pi$-periodic extension of $\phi$.*

Note that the last equality[†] relates Fourier's approach to the heat equation to d'Alembert's approach. This observation was of considerable historical importance!

**A remark on inhomogenous BVPs:** Shifting the data

There is a useful trick to replace inhomogenous boundary conditions in a problem with homogenous ones by subtracting a function satisfying those conditions.

For example if the problem is on the interval $[0,\ell]$ with constant boundary conditions $u(0,t) = a$ and $u(\ell,t) = b$, then the linear function

$$f(x) = a + \frac{b-a}{\ell}\, x$$

clearly satisfies these conditions. Set

$$v(x,t) = u(x,t) - f(x).$$

If $u$ satisfies a heat (or wave) equation with initial data $\phi$ (and $\psi$ for the velocity in the wave case) then $v$ will satisfy the same PDE (since $f$ is independent of $t$ and linear in $x$) with "shifted" initial data $\phi - f$ (and $\psi$, still, in the wave case) and homogeneous boundary data.

An example of this (for the heat equation with initial temperature zero, and with $a = 0$ and $b = 10$) is given in the homework. Make sure you can do this problem.

## 7 Laplace's Equation

Our focus above has been on the one-dimensional wave and heat equations. In higher dimensions these equations take the form $u_{tt} = c^2 \Delta u$ and $u_t = k\Delta u$, where

$$\Delta u = u_{xx} + u_{yy} + \cdots$$

is the Laplacian of $u$. Here $x, y, \ldots$ are the spatial variables (so $\Delta u = u_{xx}$ in one dimension, $\Delta u = u_{xx} + u_{yy}$ in two, and so forth). Under suitable conditions, the solutions to these equations approach a limit as $t \to \infty$. Such a time independent or "steady-state" solution to the wave or heat equation therefore satisfies Laplace's equation:

$$\boxed{\Delta u = 0}$$

Solutions to Laplace's equation are called harmonic functions. These include all linear functions, and (in dimensions $> 1$) many others such as $u(x,y) = e^x \sin y$ (check this). They

[†] proved by plugging in the Fourier sine series expansion of $\phi$ at $x \pm ct$, and then expanding each term using the trig identity for the sine of a sum

arise in many areas of mathematics and physics, including electrostatics, fluid dynamics (in the study of an irrotational flow of an incompressible fluid), gauge theory, complex function theory, minimal surface theory, the study of Brownian motion, etc.

In one dimension, Laplace's equation is simply $u_{xx} = 0$ with solutions $u(x) = A + Bx$. There's nothing more to it: The harmonic functions of one variable are exactly the linear functions. It follows that

$\exists$ unique harmonic function on any interval with prescribed boundary values

Remarkably, this fact generalizes to higher dimensions if one substitutes for the interval any bounded open region $D$ in $\mathbb{R}^n$ with sufficiently smooth boundary $\partial D$:

$\exists$ unique harmonic function on $D$ limiting to a prescribed continuous function on $\partial D$

The problem of finding such a harmonic function $u$ is called the Dirichlet problem for $D$, and can be quite difficult in general. The uniqueness of the resulting continuous function $u : \overline{D} \to \mathbb{R}$ (where $\overline{D} = D \cup \partial D$ is the closure of $D$) is an easy consequence of the following basic result:

**Maximum principle** *Any $u : \overline{D} \to \mathbb{R}$ as above (continuous on $\overline{D} \subset \mathbb{R}^n$, harmonic on $D$) attains its maximum and minimum values on the boundary $\partial D$, i.e. there exist $x_0, x_1 \in \partial D$ such that $u(x_0) \leq u(x) \leq u(x_1)$ for all $x \in D$.*

Proof Since a minimum point for $u$ is just a maximum point for $-u$ (which is also harmonic on $\overline{D}$) it is enough to show how to find a maximum point for $u$.

Let $x_1$ be any maximum point for the restriction of $u$ to $\partial D$, which exists since we are assuming $D$ is bounded ($\Longrightarrow \partial D$ is compact[†]). We claim that $x_1$ is the desired maximum point for $u$ on all of $\overline{D}$. In fact we will show that for every $\varepsilon > 0$,

$$u(x) < u(x_1) + \varepsilon s \qquad \text{for all } x \in D$$

where $s$ is the maximum squared norm of points on $\partial D$, which will complete the proof by letting $\epsilon \to 0$.

So given $\varepsilon > 0$, set $v(x) = u(x) + \varepsilon \, \|x\|^2$. Then

$$\Delta v \;=\; \Delta u + 2n\varepsilon \;=\; 0 + 2n\varepsilon \;>\; 0$$

(Note that $\Delta\|x\|^2 = \Delta(x^2 + y^2 + \cdots) = 2 + 2 + \cdots = 2n$ in $\mathbb{R}^n$.) Now let $y$ be a maximum point for $v$ on $\overline{D}$, which must in fact lie in $\partial D$ since $\Delta v$ cannot be positive at any maximum point of $v$ in $D$ (by the second derivative test in calculus). Thus for all $x \in D$ we have

$$u(x) \;\leq\; v(x) \;\leq\; v(y) \;=\; u(y) + \varepsilon\|y\|^2 \;\leq\; u(x_1) + \varepsilon s \qquad \square$$

**Corollary** (Uniqueness of the solution to the Dirichlet problem) *If $u, v : \overline{D} \to \mathbb{R}$ are solutions to the Dirichlet problem with the same boundary condition, then $u = v$.*

Proof The harmonic function $u - v$ is zero on $\partial D$, and so zero on $D$ by the maximum principle. Thus $u = v$ on $\overline{D}$. $\square$

[†]From real analysis, continuous $C \to \mathbb{R}$ with compact attains a maximum somewhere in $C$

## The Dirichlet problem for the disk

Explicit solutions to Dirichlet's problem are known for many nice regions, such as discs, annuli, wedges, squares in the plane. Here we will treat the case of discs (and their complements) which leads to Poisson's famous formula.

First observe that when one considers regions related to circles (e.g. discs, annuli and wedges) it often helps to use polar coordinates. So we begin by writing the 2-dimensional Laplacian in polar coordinates:

$$\boxed{\Delta u = u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta}} \qquad (*)$$

If you've never seen this before, study pages 150–1 in Strauss where it is carefully derived.†

Now the simplest solutions to look for are the "rotationally invariant" ones $u(r,\theta) = u(r)$ (depending only on $r$) for which Laplace's equation reduces to

$$u_{rr} + \frac{1}{r}u_r = 0 \qquad \text{or equivalently} \qquad ru_{rr} + u_r = (ru_r)_r = 0$$

Integrating we get $ru_r = a$ (a constant) $\Longrightarrow u_r = a/r$, and so integrating again gives

$$\boxed{u(r,\theta) = a\log r + b}$$

where $a$ and $b$ are constants. Of course we must have $a = 0$ if the region $D$ includes the origin (since log is undefined at 0) and so this shows in particular that the solution to the Dirichlet problem on the disc with constant boundary data is constant (which we could have guessed). If the $D$ is an annulus (the region between two concentric circles) centered at the origin, with $u$ (or $u_r$; see homework) a prescribed constant on each boundary circle, then one can readily find $a$ and $b$.

Now the general Dirichlet problem (in polar coordinates) for the disc of radius $a$

$$r^2u_{rr} + ru_r + u_{\theta\theta} = 0 \qquad \text{with} \qquad u(a,\theta) = h(\theta)$$

can be solved by separating variables, and then proceeding using Fourier series, much like the 1-dimensional heat and wave equations on an interval:

For a separated solution $u(r,\theta) = R(r)\Theta(\theta)$ (where $\Theta$ is $2\pi$-periodic) the PDE becomes

$$r^2R''\Theta + R'\Theta + R\Theta'' = 0 \quad \Longrightarrow \quad r^2R''/R + rR'/R = -\Theta''/\Theta$$

which reduces to two ODEs

$$\Theta'' = -\lambda\Theta \qquad \text{and} \qquad r^2R'' + rR' - \lambda R = 0$$

for some constant $\lambda$.

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†Similarly the 3-dimensional Laplacian is given by

$$\Delta u = u_{\rho\rho} + \frac{2}{\rho}u_r + \frac{1}{\rho^2}(u_{\phi\phi} + \cot\phi\ u_\theta + \csc^2\phi\ u_{\theta\theta})$$

as shown on page 153 of Strauss.

Solutions to the first equation (e.g. $\cos\sqrt{\lambda}\theta)$ must be $2\pi$-periodic, which force $\lambda = n^2$ for some $n = 0, 1, 2, \ldots$. If $\lambda = 0$, then $\Theta'' = 0 \Longrightarrow \Theta$ is linear and periodic, and therefore constant. For $n \geq 1$ the general solution is $\Theta(\theta) = A\cos n\theta + b\sin n\theta$, and we must also then solve the corresponding second equation

$$r^2R'' + rR' - n^2R = 0$$

This is an example of an Euler "equidimensional" equation. Rather than discuss the general approach to such equations, we simply note that $R = r^n$ is a solution.† Thus for $\lambda = n \geq 1$ we have the solutions $r^n(A\cos n\theta + B\sin n\theta)$, which suggests the following series for the general solution to the PDE

$$\boxed{u(r,\theta) = \frac{1}{2}A_0 + \sum_{n=1}^{\infty} r^n(A_n\cos n\theta + B_n\sin n\theta)}$$

where, by the boundary condition $u(a,\theta) = h(\theta)$, the coefficients are determined by Euler's formulas

$$\boxed{A_n = \frac{1}{\pi a^n}\int_0^{2\pi} h(\phi)\cos n\phi\, d\phi} \qquad \text{and} \qquad \boxed{B_n = \frac{1}{\pi a^n}\int_0^{2\pi} h(\phi)\sin n\phi\, d\phi}$$

**Example** Find a harmonic function $u(r,\theta)$ on the disk of radius 3 with boundary condition $u(3,\theta) = 2 + 5\cos\theta + 7\sin 2\theta$. Solution: Because the boundary condition is given to us as a Fourier series, we can read off the coefficients in the formula above without integrating, namely $A_0 = 4$, $A_1 = 5/3$, $B_2 = 7/9$, and the rest are zero. Thus

$$u(r,\theta) = 2 + \frac{5r}{3}\cos\theta + \frac{7r^2}{9}\sin 2\theta.$$

## Poisson's Formula

It is a remarkable fact that the infinite series in the formula above can be summed explicitly. Just plug in the integral formulas for the coefficients and use the trig identity for the cosine of a difference to get

$$\boxed{u(r,\theta) = \frac{1}{2\pi}\int_0^{2\pi} h(\phi)P_r(\theta-\phi)\, d\phi}$$

where

$$P_r(\theta) = 1 + 2\sum_{n=1}^{\infty}\left(\frac{r}{a}\right)^n \cos n\theta$$

†This is seen by direct substitution: $r^2n(n-1)r^{n-2} + rnr^{n-1} - n^2r^n = 0$. How might one arrive at this solution? Well, it is natural to try powers $r^p$ of $r$, since these will at least yield terms of equal degree, and the values of $p$ that work are the solutions to the algebraic equation $p(p-1) + p - n^2 = p^2 - n^2 = 0$, i.e. $p = \pm n$. But this already gives two linearly independent solutions, $r^n$ and $r^{-n}$, and so the general solution must be a linear combination of these. In our case we do not consider $r^{-n}$ since we want a solution on the whole disk, including its center $r = 0$.

This function $P_r(\theta)$ is called the Poisson kernel for the disk of radius $a$. It can expressed in closed form by substituting $\cos n\theta = \frac{1}{2}(e_+^n + e_-^n)$ (where for convenience we set $e_\pm = e^{\pm i\theta}$) and then summing the resulting geometric series:

$$\begin{aligned}
\boxed{P_r(\theta)} &= 1 + \sum_{n=1}^{\infty} \left(\frac{r}{a}\right)^n e_+^n + \sum_{n=1}^{\infty} \left(\frac{r}{a}\right)^n e_-^n = 1 + \frac{re_+}{a - re_+} + \frac{re_-}{a - re_-} \\
&= \frac{a^2 - ar(e_+ + e_-) + r^2 e_+ e_- + ar(e_+ + e_-) - 2r^2 e_+ e_-}{a^2 - ar(e_+ + e_-) + r^2 e_+ e_-} \\
&= \boxed{\frac{a^2 - r^2}{a^2 - 2ar\cos\theta + r^2}}
\end{aligned}$$

since $e_+ e_- = 1$ and $e_+ + e_- = 2\cos\theta$. This gives Poisson's Formula

$$\boxed{u(r,\theta) = \frac{a^2 - r^2}{2\pi} \int_0^{2\pi} \frac{h(\phi)}{a^2 - 2ar\cos(\theta - \phi) + r^2}\, d\phi}$$

for the harmonic function on the disk $D$ of radius $a$, extending $h(\theta)$ on $\partial D$.

Note the

$$u(0,0) = \frac{1}{2\pi} \int_0^{2\pi} h(\phi) d\phi$$

This is the Mean Value Property for harmonic functions: the value at the center of any disk is the average of the boundary values.

One striking consequence is that a non-constant harmonic function cannot assume its maximum and minimum values in a region $D$ except on $\partial D$.